Problem sheet 2

(1) The ergodic CLT we have stated requires ϕ-mixing. Irrational circle rotations are ergodic but not mixing. So does the CLT hold? Try simulating: Choose Y_0 uniform on $[0, 1)$, and define $Y_n = \lfloor \alpha n \rfloor$, where $\alpha = \sqrt{2} - 1$. Let $X_i = \mathbf{1}_{\{0 \le Y_i < 1/2\}}$. Compute multiple realisations of $S_n = \sum_{i=0}^{n} X_i - n/2$. Estimate the variance σ_n^2.

(2) The Galton–Watson process is the simplest sort of branching process. We start with X_0 individuals, and in each generation each individual independently has a random number of offspring, with $\mathbb{P}\{k \text{ offspring}\} = a_k$. Let X_n be the number of individuals in generation n. If $\mu = \sum k a_k$ is the expected number of offspring, show that $\mu^n X_n$ is a martingale. Use this to conclude that the population dies out (that is, eventually $X_n = 0$) almost surely if and only if $\mu \le 1$. What does this tell us about the behaviour of X_n as $n \to \infty$ when $\mu > 1$?

(3) Let $S_n = s_0 + X_1 + \cdots + X_n$ be a simple random walk with $X_i = \pm 1$ i.i.d. and s_0 a positive integer.

(a) Show that for any θ,

$$Y_n = e^{\theta S_n - n \log \cosh(\theta)}$$

is a martingale.

(b) Let $T = \min\{n : S_n = 0\}$. Use the Optional Stopping Theorem to compute the moment generating function $M_T(\lambda) = \mathbb{E}[e^{\lambda T}]$ for $\lambda < 0$.

(c) Compute $\mathbb{P}\{T < \infty\}$ and $\mathbb{E}[T]$.

(d) Do the same for the asymmetric random walk $\mathbb{P}\{X_i = +1\} = p = 1 - \mathbb{P}\{X_i = -1\}$, where $0 < p < \frac{1}{2}$.

(e) Show that

$$Z_n = (S_n + (1 - 2p)n)^2 - p(1 - p)n$$

is a martingale. Use this to compute $\operatorname{Var}(T)$.