Problem sheet 1

(1) (a) Prove that rotations of the circle $x \mapsto x + \alpha \pmod 1$ are not ergodic if α is rational.

(b) Prove that rotations of the circle are never mixing. Hint: Let α be irrational. Let $S := \{n : \lfloor n\alpha \rfloor \in [1/3, 2/3]\}$ and let A is the event $\{X_0 \in [1/4, 3/4]\}$. Show that for $n \in S$,

$$\left| \mathbb{P}\left(A \cap \tau^{-n} A\right) - \mathbb{P}(A)^2 \right| > \frac{1}{12}.$$

Infer (from the Birkhoff ergodic theorem) that S is infinite, and conclude that the sequence is not mixing.

(2) Let $X_0, X_1, \ldots$ be a stationary Markov chain. Show that

(a) it is ergodic if and only if it is irreducible and aperiodic; and

(b) when we extend it to a two-sided stationary sequence $\ldots, X_{-1}, X_0, X_1, \ldots,$ the sequence $X_0, X_{-1}, X_{-2}, \ldots$ is also a Markov chain, and find its transition probabilities.

(3) Define $X_0 \in [0, 1]$ to have the beta distribution with parameters (a, a), where $a > \frac{1}{2}$; and define $X_{n+1} = 4\xi_n X_n(1 - X_n)$, where (ξ_i) are i.i.d. with distribution $\beta(a + \frac{1}{2}, a - \frac{1}{2})$.

(a) Show that this is a stationary Markov chain.

(b) Define the Lyapunov exponent to be $\mathbb{E}[\log(4\xi_n|1 - 2X_n|)]$; this is the expected log derivative of the map taking X_n to X_{n+1}. Try simulating this for different values of a. In particular, the behaviour is very different when the Lyapunov exponent is positive (as it is for $a > 2$) and negative (for $a < 1.5$). Try to find the exact boundary.