Problem sheet 1

(1) Define $X_0 \in [0,1]$ to have the beta distribution with parameters (a,a), where $a > \frac{1}{2}$; and define $X_{n+1} = 4\xi_n X_n(1-X_n)$, where (ξ_i) are i.i.d. with distribution $\beta(a+\frac{1}{2}, a-\frac{1}{2})$.
   (a) Show that this is a stationary Markov chain.
   (b) Define the Lyapunov exponent to be $\mathbb{E}[\log(4\xi_n|1-2X_n|)]$; this is the expected log derivative of the map taking X_n to X_{n+1}. Try simulating this for different values of a. In particular, the behaviour is very different when the Lyapunov exponent is positive (as it is for $a > 2$) and negative (for $a < 1.5$). Try to find the exact boundary.

(2) The ergodic CLT we have stated requires ϕ-mixing. Irrational circle rotations are ergodic but not mixing. So does the CLT hold? Try simulating: Choose Y_0 uniform on $[0,1)$, and define $Y_n = \lfloor \alpha n \rfloor$, where $\alpha = \sqrt{2}-1$. Let $X_i = \mathbf{1}_{\{0 \le Y_i < 1/2\}}$. Compute multiple realisations of $S_n = \sum_{i=0}^{n} X_i - n/2$. Estimate the variance σ_n^2.

PROBLEM SHEET 2

(3) Let $X_1, \dots, X_n$ be i.i.d. with mean 0 and variance σ^2. $Y_n = n^{-1/2}(X_1 + \cdots + X_n)$. Show

(a) If $\mathbb{E}[e^{X_i^{1+\delta}}] < \infty$ for some $\delta > 0$ then for some constant C

$$\log \mathbb{P}\{Y_n \geq n^\alpha z\} \leq C - \frac{n^{2\alpha} z^2}{2} \tag{16}$$

for $0 \leq \alpha \leq \frac{1}{2}$.

(b) If X_i has heavy tails, so that $\mathbb{E}[|X_i|^k] = \infty$ for some positive integer k, then (16) does not hold. Try showing with simulations that the tail behaviour is really different from Gaussian, even while the core of the distribution is converging to normal.

(4) Compare Hoeffding's inequality for $\mathbb{P}\{X_1 + \cdots + X_n \geq (\mu + \epsilon)n\}$ to what you would expect from the Central Limit Theorem in the cases where the X_i are i.i.d. with mean μ and distribution

(a) Bernoulli with parameter $\frac{1}{2}$;

(b) Bernoulli with parameter $p \neq \frac{1}{2}$;

(c) Uniform on $[0, 1]$.

What about Bernstein's inequality? Can you come up with a bound that works when X_i is exponential with parameter 1?