Problem Sheet 1

Revision, lifetime distributions

1. (a) Let $L_1, \ldots, L_n$ be independent $\text{Exp}(\lambda)$ random variables. Show that the maximum likelihood estimator for λ is given by

$$\hat{\lambda} = \frac{n}{L_1 + \ldots + L_n}. \tag{1.1}$$

(b) The following data resulted from a life test of refrigerator motors (hours to burnout):

Hours to burnout				
104.3	158.7	193.7	201.3	206.2
227.8	249.1	307.8	311.5	329.6
358.5	364.3	370.4	380.5	394.6
426.2	434.1	552.6	594.0	691.5

i. Assuming refrigerator motors have $\text{Exp}(\lambda)$ lifetimes, give the maximum likelihood estimate for λ.

ii. Still assuming $\text{Exp}(\lambda)$ lifetimes, calculate the Fisher information and construct approximate 95% confidence intervals for λ and $1/\lambda$ using the approximate Normal distribution of the maximum likelihood estimator.

iii. Still assuming $\text{Exp}(\lambda)$ lifetimes, show that $2n\lambda/\hat{\lambda} \sim \chi^2_{2n}$. Let a be such that $\mathbb{P}(2n\lambda/\hat{\lambda} \leq a) = \alpha/2$ and b such that $\mathbb{P}(2n\lambda/\hat{\lambda} \geq b) = \alpha/2$. Deduce an exact 95% confidence interval for $1/\lambda$.

iv. Produce a histogram of the data and comment.

v. Merge columns of your histogram appropriately to test whether the hypothesis of $\text{Exp}(\lambda)$ lifetimes can be rejected. Use a χ^2 goodness of fit test.

2. (a) Let $T_1, \ldots, T_m$ be independent continuous nonnegative random variables with hazard functions $h_1(\cdot), \ldots, h_m(\cdot)$. Prove that $T = \min(T_1, \ldots, T_m)$ has hazard function $h_1(\cdot) + \ldots + h_m(\cdot)$.

(b) Let $T_1, \ldots T_m$ be independent random variables with Weibull distributions with rate parameters $k_1, \ldots k_m$ and common exponent n. Prove that $T = \min(T_1, \ldots, T_m)$ also has a Weibull distribution with exponent n.

(c) Calculate the hazard function of the truncated exponential distribution with maximal age ω, and calculate the limit (in distribution) as $\lambda \downarrow 0$.

3. Let λ be any positive function and $\Lambda(t) = \int_0^t \lambda(s)ds$. Suppose X is a random variable with exponential distribution with parameter 1. Show that $\Lambda^{-1}(X)$ is a random variable with hazard rate $\lambda(t)$.

4. (a) Show that for a random variable T that has as its distribution a mixture of exponential distributions $f_{T|M=\lambda}(t) = \lambda e^{-\lambda t}$, with mixing variable (random parameter) M, the unconditional mean and variance are given by

$$\mathbb{E}(T) = \mathbb{E}\left(\frac{1}{M}\right) \qquad \text{and} \qquad \text{Var}(T) = 2\mathbb{E}\left(\frac{1}{M^2}\right) - \left(\mathbb{E}\left(\frac{1}{M}\right)\right)^2, \tag{1.2}$$

and the (unconditional) survival function of T is given by

$$\bar{F}_T(t) = \mathcal{M}_M(-t), \qquad \text{where } \mathcal{M}_M(c) = \mathbb{E}(e^{cM}) \tag{1.3}$$

is the moment generating function of M.

(b) Now take as mixing distribution a Gamma distribution with parameters α and ν, i.e. $f_M(\lambda) = \nu^\alpha \lambda^{\alpha-1} e^{-\nu\lambda}/\Gamma(\alpha)$. Show that the corresponding mixture of exponential distributions has density

$$f_T(t) = \frac{\alpha\nu^\alpha}{(t+\nu)^{\alpha+1}} \tag{1.4}$$

Also calculate the survival function and hazard rate.

5. (a) Show that $\mathbb{E}(d_x - q_x\ell_x) = 0$ and $\text{Var}(d_x - q_x\ell_x) = q_x(1-q_x)\mathbb{E}(\ell_x)$. Hint: Condition on ℓ_x. What is the conditional distribution of d_x given ℓ_x?

   (b) Is $\hat{q}_0^{(0)} = d_0/\ell_0$ unbiased? What about $\hat{q}_1^{(0)} = d_1/\ell_1$? Calculate the (approximate) Fisher Information matrix, the (approximate) variances of $\hat{q}_0^{(0)}$ and $\hat{q}_1^{(0)}$, and their estimates induced by the maximum likelihood estimates of $\hat{q}_x^{(0)}$.

6. The survival times (in days after transplant) for the original $n = 69$ members of the Stanford Heart Transplant Program were as follows:

Survival time after heart transplant (days)							
15	3	624	46	127	64	1350	280
23	10	1024	39	730	136	1775	1
836	60	1536	1549	54	47	51	1367
1264	44	994	51	1106	897	253	147
51	875	322	838	65	815	551	66
228	65	660	25	589	592	63	12
499	305	29	456	439	48	297	389
50	339	68	26	30	237	161	14
167	110	13	1	1			

The aim of this exercise is to construct the associated lifetable.

(a) Complete the following table of counts d_x of associated curtate residual lifetimes (in years=365 days), counts ℓ_x of subjects alive exactly x years after their transplant, total time $\tilde{\ell}_x$ spent alive between x and $x+1$ years after their transplant, by all subjects:

x	0	1	2	3	4
d_x			8	4	3
ℓ_x					
$\tilde{\ell}_x$		19.148	10.203	4.937	1.315

(b) Calculate the maximum likelihood estimators $\hat{q}_x^{(0)}$ and $\hat{q}_x$ for q_x, $x = 0, \ldots, 4$, based on the discrete and continuous method, respectively.

(c) Calculate the maximum likelihood estimates. Comment on the differences.

(d) Estimate the probability to survive for 3 months

i. assuming fractional and integer parts of lifetimes are independent, and the fractional part is uniform;

ii. assuming the force of mortality is constant over the first year;

iii. directly from the data (the total time spent alive until three months after the transplant is 12.584 years). Hint: You may, of course, guess formulas to test your intuition, but you should then state your assumptions and apply the discrete and/or continuous method to justify your estimates as maximum likelihood estimates.