Instability of Baroclinic Waves with Bottom Slope*

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ABSTRACT

Pedlosky's theory explaining the behavior of unstable baroclinic waves in the β-plane is modified to include a sloped bottom (although the β effect is ignored). The result found is the same sort of nonlinear oscillatory behavior described by Pedlosky, except in the case of short wavelengths for negative shears. In that case, the theory predicts an initial explosive growth of the wave amplitude, so that it will reach amplitudes that are very large compared with its initial scale. This suggests a possible mechanism for small-scale current fluctuations in the oceans.

1. Introduction

In an earlier paper of Pedlosky (1970), a perturbation analysis was used to examine the effects of nonlinear terms on the stability of baroclinic waves in a β-plane system. The essential feature of this problem was the critical shear curve shown in Fig. 1. Let k be the wavenumber in the direction of flow scaled by the channel width in which the shear flow is contained. The similarly scaled y wavenumber, l, is a multiple of π (see section 2). When $a^2 \equiv k^2 + l^2 > 2F$, where F is the square of the ratio of L to the deformation radius (see section 2), the wave is always stable. For $a^2 < 2F$, the wave is unstable for shears larger than the critical shear; the linearized equations predict an exponentially growing amplitude in this region. In his analysis, Pedlosky took a shear slightly larger than the critical value, U_c, to examine the effect of the nonlinearity. Intuitively, we would expect the growing wave to remove energy from the shear, thus decreasing the shear below the critical value, since it was only slightly above to begin with. Being an action of the perturbation on itself, this is a second-order effect and the nonlinearity might be thought to stabilize the amplitude. The growth of the wave should taper off rather than growing indefinitely, and, in fact, this is what he found.

If we consider a case in which the bottom of the channel is sloped, the character of the problem changes. Instead of the critical shear curve shown in Fig. 1, we get the result shown in Fig. 2. Note that the complete solution for the marginal curve which yields U_c as a function of wavenumber, with β and bottom slope (η) both nonzero, would have terms which are multiples both of η and of β. In Pedlosky's problem, $\eta = 0$, so U_c was only a multiple of β. In this problem, U_c is a multiple of η. If β and η were *both* 0, of course, U_c would vanish, and the problem would be meaningless. Our problem substitutes η for β. They are very similar in their effects. The only difference is that β acts on both layers equally, while η acts only on the bottom layer. If we were to have both top slope *and* bottom slope, each of slope η, our Eq. (3.7) would take exactly the same form as Pedlosky's, only with η substituted for β. It is this that creates the asymmetry between positive and negative critical shear which is not present in Pedlosky's case. Here we do not automatically have stability for $a^2 > 2F$. Instead, for negative shears, there is a region of instability for arbitrarily large wavenumbers. Furthermore, on the lower branch, for $a^2 > 2F$, a decrease in the shear actually pushes the system further into the unstable shear region. The result implying nonlinear stabilization in Pedlosky's paper cannot a priori be assumed valid for this problem.

To understand the effect of the bottom slope on the wave growth and, in particular, to understand the behavior on that inverted part of the curve, the methods of Pedlosky are applied to a new calculation which includes bottom slope.

2. The model

In the present model there are two layers of homogeneous fluid with slightly different constant densities confined to a channel which extends infinitely both ways in the east–west direction (x' axis) and has constant width L in the north–south direction (y' axis) (bounded by solid vertical walls). The up direction is the (z' axis). The channel rotates with constant angular velocity, Ω. The two layers are assumed to have equal average height, $H/2$. The interface is at a height h' above

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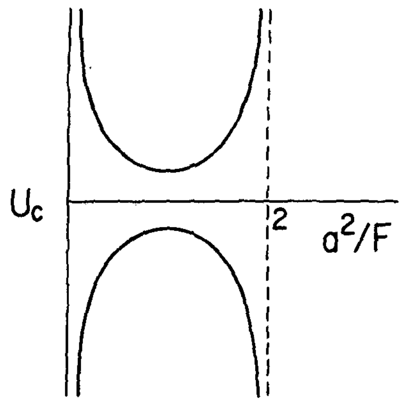

FIG. 1. A schematic of the two-layer neutral stability curve in the absence of bottom slope. The layer depths are equal. Note the symmetry for positive and negative shear.

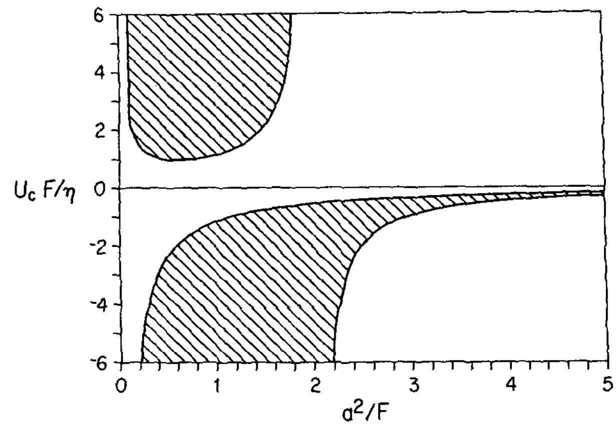

FIG. 2. The marginal stability curve in the presence of topography and $\beta = 0$. Note the thin region of instability at high wavenumber for $U_c\eta < 0$.

the $z' = H/2$ plane. The bottom of the channel has a constant slope η' in the y' direction. The velocity components along the x', y', z' axes are u'_k, v'_k, w'_k, respectively, where $k = 1$ or 2 depending on whether it is the velocity in the upper or lower layer. The U is a characteristic scale for the horizontal velocity, and pressure in layer k is p_k with density ρ_k (see Fig. 3).

The planetary vorticity gradient, β, is assumed to be negligible, as are frictional effects. This is especially valid for small zone wavelengths.

We use the following nondimensional variables:

$$(x, y) = (x', y')/L$$

$$z = z'/D$$

$$(u_k, v_k) = (u'_k \cdot v'_k)/U$$

$$w_k = (L/UH)w'_k$$

$$t = (U/L)t'$$

$$\psi_k = \frac{p_k}{(2\Omega)\rho_k UL}$$

$$h = h' \frac{g(\rho_2 - \rho_1)}{\rho_1 UL(2\Omega)}$$

$$\epsilon = U/(2\Omega)L = \text{Rossby Number}$$

$$g' = (\rho_2 - \rho_1)/\rho_2 g$$

$$F = \frac{(2\Omega)^2 L^2}{g'(H/2)}$$

$$\eta = \frac{L\eta'}{(H/2)\epsilon}.$$

3. The equations

If we assume $\epsilon \ll 1$, we can use the geostrophic approximation to write the horizontal velocities (Pedlosky, 1970) as

$$u_k = -\frac{\partial \psi_k}{\partial y}$$

$$v_k = \frac{\partial \psi_k}{\partial x}. \tag{3.1}$$

Note that, by the Taylor–Proudman theorem (Pedlosky, 1979, pp. 43–45), we may assume the horizontal velocities to be independent of height within each layer. We can then write the conservation of potential vorticity in each layer as (Pedlosky, 1970)

$$\frac{\partial q_k}{\partial t} + J(\psi_k, q_k) = 0 \tag{3.2}$$

where

$$q_1 = \nabla^2 \psi_1 + F(\psi_2 - \psi_1)$$

$$q_2 = \nabla^2 \psi_2 + F(\psi_1 - \psi_2) + \eta. \tag{3.3}$$

We have the boundary conditions

$$\frac{\partial \psi_k}{\partial x} = 0 \quad \text{at} \quad y = 0, 1 \tag{3.4}$$

$$\lim_{X \to \infty} \frac{1}{X} \int_{-X}^{X} \frac{\partial^2 \psi_k}{\partial y \partial t} dx = 0 \quad \text{at} \quad y = 0, 1 \tag{3.5}$$

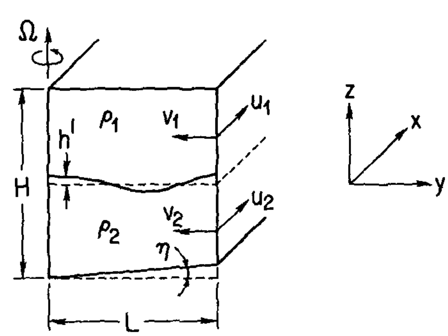

FIG. 3. A schematic of the physical model.

resulting from the restriction that there is no flow through the walls of the channel.

We suppose that there is a basic flow which is a constant velocity U_k in the x direction. This corresponds to a streamfunction

$$\psi_k^{(0)} = -U_k y. \tag{3.6}$$

Upon this we superimpose a small perturbation,

$$\psi_k = \psi_k^{(0)} + \phi_k.$$

Equation (3.2) may then be written as

$$\begin{aligned}0 &= \left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x}\right)[\nabla^2\phi_1 + F(\phi_2-\phi_1)] + \frac{\partial\phi_1}{\partial x}F(U_1-U_2)\\ &\quad + J[\phi_1, \nabla^2\phi_1 + F(\phi_2-\phi_1)]\\ 0 &= \left(\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right)[\nabla^2\phi_2 + F(\phi_1-\phi_2)]\\ &\quad + \frac{\partial\phi_2}{\partial x}[F(U_2-U_1)+\eta] + J[\phi_2, \nabla^2\phi_2 + F(\phi_1-\phi_2)].\end{aligned} \tag{3.7}$$

Note that the Jacobians here are quadratic in ϕ, while all the other terms are first order.

4. The linear problem

If the disturbances are very small, the quadratic terms will be negligible. If we then neglect them, we get the linear equations

$$\begin{aligned}0 &= \left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x}\right)[\nabla^2\phi_1 + F(\phi_2-\phi_1)] + \frac{\partial\phi_1}{\partial x}F(U_1-U_2)\\ 0 &= \left(\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right)[\nabla^2\phi_2 + F(\phi_1-\phi_2)]\\ &\quad + \frac{\partial\phi_2}{\partial x}[F(U_2-U_1)+\eta].\end{aligned} \tag{4.1}$$

These equations (and the boundary conditions) have solutions of the form

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = R_e\begin{pmatrix}1\\ \gamma\end{pmatrix}A_e^{i\alpha(x-ct)}\sin(m\pi y) \tag{4.2}$$

where m is an integer, α is a positive real number, and A, γ and c are complex. The A and γA are amplitudes in the upper and lower layers, respectively.

Using (4.1), we can find γ and c:

$$c = \frac{U_1+U_2}{2} - \frac{\eta}{2a^2}\cdot\frac{a^2+F}{a^2+2F} \pm \frac{\sqrt{D}}{2a^4+4a^2F} \tag{4.3}$$

where

$$\begin{aligned}D &= (U_1-U_2)^2a^4(a^4-4F^2)\\ &\quad + 2(U_1-U_2)\eta(a^6+a^4F-2F^2a^2) + \eta^2(a^2+F)^2\end{aligned} \tag{4.4}$$

$$\gamma = \frac{a^2}{F} + \frac{U_2-c}{U_1-c} = \frac{a^2}{F} + 1 - \frac{U_1-U_2}{U_1-c} \quad (a^2 = \alpha^2 + m^2\pi^2). \tag{4.5}$$

Let $c = c_r \pm ic_i$ (c_r, c_i real). Then, letting $\|\phi\|$ be the amplitude of ϕ's fluctuation,

$$\begin{aligned}\|\phi_1\| &= \|A\sin(m\pi y)\|\,\|e^{i\alpha(x-ct)}\|\\ &= |A|\,|e^{i\alpha(x-c_rt)}|\cdot|e^{\pm c_it}|\\ &= e^{\pm\alpha c_it}|A|.\end{aligned} \tag{4.6}$$

If $c_i = 0$ (i.e., $D \geqslant 0$), the amplitude of the wave remains constant, and our neglect of the quadratic terms remains valid. If, however, $c_i \neq 0$ ($D < 0$), then the wave will have one mode whose amplitude grows exponentially. Thus, the shear for which $D = 0$ is critical. Letting U_c be this critical shear,

$$U_c = \frac{\eta}{F}\cdot\frac{1}{x(x-2)}\left[1 - x \pm \frac{2}{\sqrt{x+2}}\right], \quad x = \frac{a^2}{F} \tag{4.7}$$

which is shown in Fig. 2.

The curve has two branches (one of which is itself split) corresponding to the choice of sign in the equation for U_c. The one for which the minus sign is chosen (the one with the discontinuity at $x = 2$) will be called the minus branch, the other will be called the plus branch.

For shears in the shaded region, the wave amplitude grows. The linear theory predicts that the amplitude will continue growing exponentially. Of course, this is unphysical. As the wave grows, it will eventually become too large for our linear approximation to be valid. As already mentioned, one would expect the growing wave to decrease the shear appreciably if it grows large, thus affecting the stability.

5. The nonlinear approximation

To get an idea of how the nonlinearity affects the amplitude growth, we assume the shear to be only slightly above or below the critical shear, $U_1 - U_2 = U_c + \Delta$, where $\Delta \ll 1$. We then do a perturbation analysis in this parameter. From Eqs. (4.3) and (4.4), we see that

$$c_i \sim |\Delta|^{1/2}\frac{\sqrt{\eta}}{a}\left(\frac{F}{a^2+2F}\right)^{3/4}, \quad \Delta \ll 1. \tag{5.1}$$

Since the time scale for the amplitude growth predicted by the linear theory is $1/\alpha c_i$, we consider a separate, slower time variable T, defined by

$$T = |\Delta|^{1/2}t. \tag{5.2}$$

The old $(\partial/\partial t)$ now becomes

$$\frac{\partial}{\partial t} + |\Delta|^{1/2}\frac{\partial}{\partial T} \tag{5.3}$$

and Eq. (4.1) becomes

$$\left[\frac{\partial}{\partial t}+|\Delta|^{1/2}\frac{\partial}{\partial T}+(U_2+U_c+\Delta)\frac{\partial}{\partial x}\right][\nabla^2\phi_1+F(\phi_2-\phi_1)]$$
$$+\frac{\partial\phi_1}{\partial x}F(U_c+\Delta)+J[\phi_1,\nabla^2\phi_1+F(\phi_2-\phi_1)]=0$$
$$\left[\frac{\partial}{\partial t}+|\Delta|^{1/2}\frac{\partial}{\partial T}+U_2\frac{\partial}{\partial x}\right][\nabla^2\phi_2+F(\phi_1-\phi_2)]$$
$$+\frac{\partial\phi_2}{\partial x}[-F(U_c+\Delta)+\eta]$$
$$+J[\phi_2,\nabla^2\phi_2+F(\phi_1-\phi_2)]=0. \quad (5.4)$$

We can also expand ϕ_k as

$$\phi_k=|\Delta|^{1/2}\phi_k^{(1)}+|\Delta|\phi_k^{(2)}+|\Delta|^{3/2}\phi_k^{(3)}+\cdots. \quad (5.5)$$

By collecting terms of a given order in $|\Delta|^{1/2}$, we get a series of equations for the $\phi_k^{(i)}$. The full calculation is done in the Appendix.

At order $|\Delta|^{1/2}$, we get the linear result, with U_2-U_1 being U_c. At order $|\Delta|$, we get a first-order phase shift between the waves in the two layers, as well as a modification of the zonal flow. (It is the latter which reflects the "nonlinear" change in the shear discussed earlier and which, in Pedlosky's problem, stabilized the growth of the wave amplitude.) Finally, at order $|\Delta|^{3/2}$, we obtain for the first-order wave amplitude $A(T)$,

$$\frac{d^2A}{dT^2}-c_i^2\alpha^2A+N\alpha^2A[|A|^2-A(0)^2]=0 \quad (5.6)$$

where the coefficients are given by Eqs. (A28) and (A29). The N on each branch is shown in Figs. 4 and 5. Where there are two signs, the upper refers to the plus branch, the lower to the minus branch of U_c. Note that since m is a nonzero integer, $a^2 \geq \pi^2$, and the graphs are cut off at $x=\pi^2/F$.

The limit of N as $x\to\infty$ is

$$\pm\left(\frac{m^2\pi^2}{4(2m^2\pi^2+F)}+\frac{2\sqrt{2F}m\pi\tanh\sqrt{F/2}}{(2m^2\pi^2+F)^2}\right)\sqrt{x}. \quad (5.7)$$

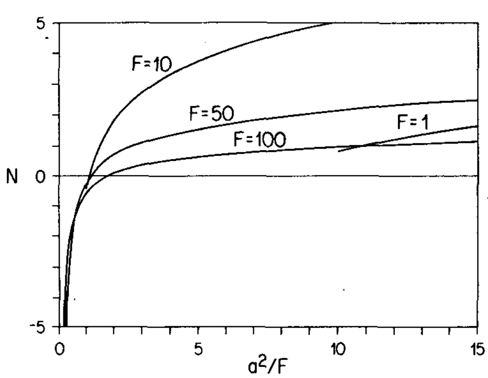

FIG. 4. The nonlinear equilibration constant, N, for the (upper) plus branch ($U_c\eta<0$).

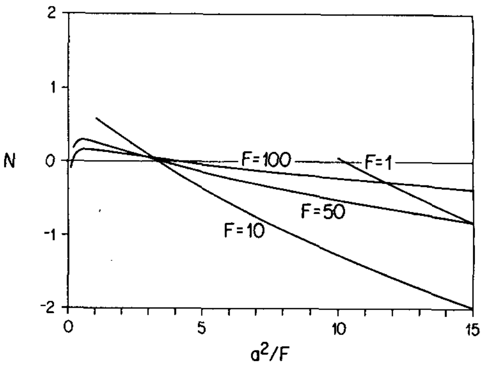

FIG. 5. The nonlinear equilibration constant, N, for the (lower) minus branch ($U_c\eta<0$).

Letting N^s be the part of N that comes from the vorticity gradient terms, we have

$$N^s=\frac{m^4\pi^4}{2m^2\pi^2+F}\left[\frac{1}{2}-\frac{\sqrt{2F}\tanh\sqrt{F/2}}{2m^2\pi^2+F}\right]\times\left[-\frac{3}{x}\pm\frac{2\sqrt{x+2}}{x}\right]. \quad (5.8)$$

This goes to zero for large x, but dominates for small x. The first bracketed term is always positive. Thus, $N^s<0$ on the plus branch, and $N^s>0$ on the minus branch for all $x>1/4$.

6. Interpretation

First, for very small A (or, equivalently, $N=0$), Eq. (A27) reduces to

$$\frac{d^2A}{dT^2}=c_i^2\alpha^2A \quad (6.1)$$

which has the same solutions as the linear problem (4.6). Note that c_i is the same as the value that was determined in (5.1) from the linear problem (except for the factor of $|\Delta|$, because of the different time scale).

If $N\neq 0$, as the wave amplitude grows, the cubic term in A will quickly become important. Thus, if $N>0$, this term will act to decrease d^2A/dt^2, slowing and eventually reversing the growth of the wave. When A has been reduced below its initial value, this term will act oppositely, to increase it. Thus, the effect of the nonlinearity is to stabilize the previously exponential growth, to cause an oscillatory behavior which has been analyzed by Pedlosky (1970).

If, on the other hand, $N<0$, then the cubic term acts to increase the growth rate. Thus, we get explosive growth of the amplitude. What this means, of course, is that the wave will quickly grow beyond its initial $O(|\Delta|^{1/2})$ level, at which point our perturbation analysis breaks down.

If we now look at the graphs, we find that on the plus branch, for sufficiently large wave numbers, $N>0$. Similarly, on the minus branch, for $a^2<2F$, except at the very smallest wave numbers, $N > 0$. (This is the part of the critical shear curve which most nearly corresponds to the curve that Pedlosky analyzed.) This seems intuitively correct, since on these curves our intuitive argument that the increasing amplitude decreases the shear below the critical value holds true, and we would expect the nonlinear term to be stabilizing.

On the minus branch for $a^2 > 2F$, however, as has already been noted, a decrease in the shear pushes it farther into the unstable region, so we would expect explosive instability. That is what we get, at least for sufficiently large wavenumbers.

What scars this intuitive argument is the effect of the changing vorticity gradient. At small wavenumbers, where N^s dominates, it acts to destabilize the plus branch and to stabilize the minus branch (except for $x < 1/4$, where it destabilizes the minus branch as well). For large x, though, N is positive on the plus branch and negative on the minus branch, as expected.

It is most interesting to note that for some wavenumbers the shear can be decreased to a point the linear theory predicts to be more unstable than the starting point, yet be stabilized by the change in the vorticity gradient. The shear can also be decreased to a point the linear theory predicts to be stable, yet the wave can grow explosively.

7. Conclusions

The results of this analysis provide a mechanism for the sudden production of very short large-amplitude waves. It is interesting that the trigger for this explosive wave growth is a *decrease* in the shear.

Since the linear theory growth rate (the rate which determines the time scale for the initial development of the wave) includes a factor of α^2 for a given total wavenumber, the growth rate is fastest for y wavenumber ($m = 1$). This effect is appreciable, since $a^2 = \alpha^2 + \pi^2 m^2$, which means that an increase of m from 1 to 2 means a decrease of α^2 by about 30 for a given total wavenumber. (Note that the unstable wavenumber range is tuned to the shear; while the range is not necessarily narrow, it is limited.) Thus, we would expect a wave that resulted from this theory to be dominated by modes with north–south wavelengths that are long relative to the east–west wavelengths.

The theory also predicts by Eq. (A4) that for large wavenumbers, the amplitude of the lower-layer wave is considerably larger than that of the upper-layer wave, and $A_1/A_2 \approx F/a^2 \to 0$ as $a^2 \to \infty$.

It seems that the only essential ingredient for this behavior is that the neutral curve have a "beak"; that is, that there are two critical values of the shear, with the unstable region lying between them. This cannot be said with certainty, since the relative importance of the shear and vorticity gradient terms may change. Furthermore, there are numerous questionable assumptions in the model, especially the neglect of the β effect and the fiat that the two layers have equal depth.

Nonetheless, there are some data to which these results may be applicable, namely, in measurements of Gulf Stream currents across the continental rise in the western North Atlantic (Luyten, 1977). Currents were measured in a region which is a fair approximation to our model; there is a steady north–south bottom slope and an east–west current. The westward flow decreases with increasing depth which means, in our notation, that the vertical shear is negative, which should be a necessary condition for instability.

The results of the measurements along the "Lower Rise" match the broad outlines of the theory's predictions. [The "Upper Rise", the region of the 4000 m isobath (depth < 4000 m), has a depth too small relative to the bottom slope, so the flow just follows the isobaths.] Mean fluctuations were found to have a zonal scale shorter than the 50 km resolution of the array, while the meridional scale was about 150 km. Furthermore, the fluctuation field was found to be bottom intensified.

This model then elucidates a mechanism which may be important in explaining the remarkably small scale of the zonal fluctuations measured by Luyten.

Acknowledgments. This work would have been impossible without help from Joe Pedlosky and Bill Young. The research was supported in part by a fellowship from Woods Hole Oceanographic Institution and by National Science Foundation's Office of Atmospheric Science.

APPENDIX A

Calculation of Coefficients in the Wave Amplitude Equation

The sequence of problems obtained by inserting (5.5) into (5.4) is similar to that found in Pedlosky (1970).

First, for $O(|\Delta|^{1/2})$, we get back the linear problem with $U_1 - U_2 = U_c$. That is,

$$\left[\frac{\partial}{\partial t} + (U_2 + U_c)\frac{\partial}{\partial x}\right][\nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})] + \frac{\partial \phi_1^{(1)}}{\partial x}FU_c = 0$$

$$\left(\frac{\partial}{\partial t} + U_2\right)\frac{\partial}{\partial x}[\nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})] + \frac{\partial \phi_2^{(1)}}{\partial x}(-FU_c + \eta) = 0. \quad \text{(A1)}$$

We can find wave solutions of the form

$$\phi_1^{(1)} = R_e A(T) e^{i\alpha(x-ct)} \sin(m\pi y)$$
$$\phi_2^{(1)} = R_e \gamma A(T) e^{i\alpha(x-ct)} \sin(m\pi y) \quad \text{(A2)}$$

where

$$c = U_2 + \frac{U_c}{2} - \frac{\eta}{2a^2} \cdot \frac{a^2+F}{a^2+2F} \quad \text{(A3)}$$

$$\gamma = 1 + \frac{a^2}{F} - \frac{U_c}{U_2+U_c-c} = \frac{(U_2-c)F}{(U_2-c)(a^2+F)+FU_c-\eta}. \quad \text{(A4)}$$

At $\alpha(|\Delta|)$ we get

$$\left[\frac{\partial}{\partial t} + (U_2+U_c)\frac{\partial}{\partial x}\right][\nabla^2\phi_1^{(2)} + F(\phi_2^{(2)} - \phi_1^{(2)})] + \frac{\partial \phi_1^{(2)}}{\partial x}FU_c + \frac{\partial}{\partial T}[\nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})] + J[\phi_1^{(1)}, \nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})] = 0$$

$$\left(\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right)[\nabla^2\phi_2^{(2)} + F(\phi_1^{(2)} - \phi_2^{(2)})] + \frac{\partial \phi_2^{(2)}}{\partial x}(-FU_c+\eta) + \frac{\partial}{\partial T}[\nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})] + J[\phi_2^{(1)}, \nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})] = 0. \quad \text{(A5)}$$

Since $\phi_1^{(1)}$, $\nabla^2\phi_1^{(1)}$, $\phi_2^{(1)}$, $\nabla^2\phi_2^{(1)}$ all differ only by a real factor, the Jacobians vanish. We then have

$$\left[\frac{\partial}{\partial t} + (U_2+U_c)\frac{\partial}{\partial x}\right][\nabla^2\phi_1^{(2)} + F(\phi_2^{(2)} - \phi_1^{(2)})] + \frac{\partial \phi_1^{(2)}}{\partial x}FU_c = R_e\left[(a^2+F)\frac{dA}{dT} - F\gamma\frac{dA}{dT}\right]e^{i\alpha(x-ct)} \times \sin(m\pi y)$$

$$\left(\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right)[\nabla^2\phi_2^{(2)} + F(\phi_1^{(2)} - \phi_2^{(2)})] + \frac{\partial \phi_2^{(2)}}{\partial x}(-FU_c+\eta) = R_e\left[(a^2+F)\gamma\frac{dA}{dT} - F\frac{dA}{dT}\right]e^{i\alpha(x-ct)}\sin(m\pi y). \quad \text{(A6)}$$

Let us look at solutions of the form

$$\phi_k^{(2)} = R_e A_k^{(2)} e^{i\alpha(x-ct)}\sin(m\pi y). \quad \text{(A7)}$$

On substituting into (A6) we get

$$-\gamma A_1^{(2)} + A_2^{(2)} = \frac{1}{i\alpha F}\frac{dA}{dT}\left[\frac{FU_c}{(U_2+U_c-c)^2}\right]$$

$$\gamma A_1^{(2)} - A_2^{(2)} = \frac{1}{i\alpha F}\frac{dA}{dT}\gamma^2\left[\frac{\eta - FU_c}{(U_2-c)^2}\right]. \quad \text{(A8)}$$

Combining these two equations, we get

$$\frac{dA}{dT}\left[\frac{FU_c}{(U_2+U_c-c)^2} - \gamma^2\frac{(FU_c-n)}{(U_2-c)^2}\right] = 0. \quad \text{(A9)}$$

Direct substitution from Eqs. (A3), (A4), and (A7) confirms that the bracketed term vanishes identically. We then get

$$A_2^{(2)} = \gamma A_1^{(2)} + \left(\frac{1}{i\alpha F}\right)\frac{dA}{dT}\left[\frac{FU_c}{(U_2+U_c-c)^2}\right]. \quad \text{(A10)}$$

We may set $A_1^{(2)} = 0$ since we assume all the structure of the $U_1 - U_2 = U_c$ wave is in the $O(|\Delta|^{1/2})$ component. Equation (A10) then just gives us a phase shift between the two layers.

To the solution already found, we may add an arbitrary homogeneous solution

$$\phi_k^{(2)} = \Phi_k^{(2)}(y, T). \quad \text{(A11)}$$

This is a zonal flow correction. To this order, then, the streamfunction is

$$\phi_1 = |\Delta|^{1/2}R_e A(T)e^{i\alpha(x-ct)}\sin(m\pi y) + |\Delta|\Phi_1^{(2)}(y,T)$$

$$\phi_2 = |\Delta|^{1/2}\gamma R_e A(T)e^{i\alpha(x-ct)}\sin(m\pi y) + |\Delta|\left[\Phi_2^{(2)}(y,T) + R_e e^{i\alpha(x-ct)} \times \sin(m\pi y)\left(\frac{1}{i\alpha}\right)\frac{dA}{dT}\left(\frac{U_c}{(U_2+U_1-c)^2}\right)\right]. \quad \text{(A12)}$$

Next we collect terms of order $|\Delta|^{3/2}$

$$0 = \left[\frac{\partial}{\partial t} + (U_2+U_c)\frac{\partial}{\partial x}\right][\nabla^2\phi_1^{(3)} + F(\phi_2^{(3)} - \phi_1^{(3)})] + \frac{\partial\phi_1^{(3)}}{\partial x}FU_c + \frac{\partial}{\partial T}[\nabla^2\phi_1^{(2)} + F(\phi_2^{(2)} - \phi_1^{(2)})] + \frac{\Delta}{|\Delta|}\left(\frac{\partial}{\partial x}[\nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})] + F\frac{\partial\phi_1^{(1)}}{\partial x}\right) + J[\phi_1^{(1)}, \nabla^2\phi_1^{(2)} + F(\phi_2^{(2)} - \phi_1^{(2)})] + J[\phi_1^{(2)}, \nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})]$$

$$0 = \left[\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right][\nabla^2\phi_2^{(3)} + F(\phi_1^{(3)} - \phi_2^{(3)})] + \frac{\partial\phi_2^{(3)}}{\partial x}(\eta - FU_c) + \frac{\partial}{\partial T}[\nabla^2\phi_2^{(2)} + F(\phi_1^{(2)} - \phi_2^{(2)})] - \frac{\Delta}{|\Delta|}F\frac{\partial\phi_2^{(1)}}{\partial x} + J[\phi_2^{(1)}, \nabla^2\phi_2^{(2)} + F(\phi_1^{(2)} - \phi_2^{(2)})] + J[\phi_2^{(2)}, \nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})]. \quad \text{(A13)}$$

Using our previous results, this becomes

$$\left[\frac{\partial}{\partial t} + (U_2+U_c)\frac{\partial}{\partial x}\right][\nabla^2\phi_1^{(3)} + F(\phi_2^{(3)} - \phi_1^{(3)})] + \frac{\partial\phi_1^{(3)}}{\partial x}FU_c = -\frac{\partial}{\partial T}\left[\frac{\partial^2\Phi_1^{(2)}}{\partial y^2} + F(\Phi_2^{(2)} - \Phi_1^{(2)})\right] + \frac{Fm\pi U_c}{4(U_2+U_c-c)^2}\sin(2m\pi y)\frac{d|A|^2}{dT}$$

$$+R_e i\alpha A e^{i\alpha(x-ct)}\sin(m\pi y)\left[-\frac{\Delta}{|\Delta|}\left(F-\frac{FU_c}{U_2+U_c-c}\right)\right.$$
$$-\frac{\partial}{\partial y}\left(\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}\right)+F(\Phi_2^{(2)}-\Phi_1^{(2)})$$
$$\left.+\frac{1}{\alpha^2 A}\frac{d^2A}{dT^2}\left(\frac{FU_c}{(U_2+U_c-c)^2}\right)-\frac{\partial\Phi_1^{(2)}}{\partial y}\left(\frac{FU_c}{U_2+U_c-c}\right)\right]$$

$$\left[\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right][\nabla^2\phi_2^{(3)}+F(\phi_1^{(3)}-\phi_2^{(3)})]$$
$$+\frac{\partial\phi_2^{(3)}}{\partial x}(\eta-FU_c)=-\frac{\partial}{\partial T}\left[\frac{\partial^2\Phi_2^{(2)}}{\partial y^2}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right]$$
$$-m\pi\frac{FU_c}{4(U_2+U_c-c)^2}\sin(2m\pi y)\frac{d|A|^2}{dT}$$
$$+R_e i\alpha e^{i\alpha(x-ct)}\sin(m\pi y)\left(\gamma F\frac{\Delta}{|\Delta|}-\gamma\frac{\partial}{\partial y}\right.$$
$$\times\left[\frac{\partial^2\Phi_2^{(2)}}{\partial y^2}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right]-(a^2+F)\frac{1}{\alpha^2 A}$$
$$\left.\times\frac{U_c}{(U_2+U_c-c)^2}\frac{d^2A}{dT^2}+\gamma\frac{FU_c-\eta}{U_2-c}\frac{\partial\Phi_2^{(2)}}{\partial y}\right). \quad \text{(A14)}$$

If the part that is independent of x and t is nonzero, then we will get solutions which have a term varying linearly with t. Thus, after a time on the order of $|\Delta|^{-1/2}$, the third-order solution will be as large as the second-order solution, contradicting the assumption implicit in our perturbation analysis that the portion of a given order is small relative to the previous order. (That is, the coefficient of any $|\Delta|^{1/2}$ is much smaller than $|\Delta|^{-1/2}$.) Thus, that part of the inhomogeneity must be 0, giving us the equations

$$0=-\frac{\partial}{\partial T}\left[\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}+F(\Phi_2^{(2)}-\Phi_1^{(2)})\right]+\frac{Fm\pi U_c}{4(U_2+U_c-c)^2}\sin(2m\pi y)\frac{d|A|^2}{dT}$$
$$0=-\frac{\partial}{\partial T}\left[\frac{\partial^2\Phi_2^{(2)}}{\partial y^2}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right]-\frac{Fm\pi U_c}{4(U_2+U_c-c)^2}\sin(2m\pi y)\frac{d|A|^2}{dT}. \quad \text{(A15)}$$

The solutions of (5.20) that satisfy the boundary conditions and are zero at $T=0$ are

$$\Phi_1^{(2)}=-\Phi_2^{(2)}=-\frac{[|A|^2-|A(0)|^2]}{8(2m^2\pi^2+F)}\frac{FU_c}{(U_2+U_c-c)^2}m\pi\times\left[\sin(2m\pi y)-\frac{\sinh[\sqrt{2F}(y-½)]}{\cosh\sqrt{F/2}}\left(\frac{m\pi}{\sqrt{F/2}}\right)\right] \quad \text{(A16)}$$

which gives us a correction to the vertical shear of

$$U_1^{(2)}-U_2^{(2)}=\frac{\partial}{\partial y}\left(\Phi_2^{(2)}-\Phi_1^{(2)}\right)=\frac{[|A|-|A(0)|^2]}{2(2m^2\pi^2+F)}\frac{m^2\pi^2FU_c}{(U_2+U_c-c)^2}\times\left[\cos(2m\pi y)-\frac{\cosh[\sqrt{2F}(y-½)]}{\cosh\sqrt{F/2}}\right]. \quad \text{(A17)}$$

If we integrate over the width of the channel, we get the average change in the shear

$$\overline{U_1^{(2)}-U_2^{(2)}}=(\Phi_2^{(2)}-\Phi_1^{(2)})|_0^1=-\frac{[|A|^2-|A(0)|^2]}{4(2m^2\pi^2+F)}\frac{Fm^2\pi^2\tanh\sqrt{F/2}}{(U_2+U_c-c)^2\sqrt{F/2}}U_c. \quad \text{(A18)}$$

Thus, the average change in shear is always opposite to the initial shear when the wave amplitude increases. An increase in amplitude decreases the shear, and a decrease in amplitude increases the shear. This is not directly relevant to the stability determination, however, since the change in shear enters the amplitude equation only when multiplied by $\sin^2(m\pi y)$ (see below).

Let us look now at solutions to the remaining portion of Eq. (A14) which are of the form

$$\phi_1^{(3)}=A_1^{(3)}\sin(m\pi y)e^{i\alpha(x-ct)}$$
$$\phi_2^{(3)}=A_2^{(3)}\sin(m\pi y)e^{i\alpha(x-ct)}. \quad \text{(A19)}$$

If we substitute this into (A14), multiply by $2\sin(m\pi y)$, and integrate from $y=0$ to 1 [that is, take the Fourier coefficient of $\sin(m\pi y)$], we get

$$-F(U_2+U_c-c)\gamma A_1^{(3)}+F(U_2+U_c-c)A_2^{(3)}$$
$$=-\frac{\Delta}{|\Delta|}\frac{U_2-c}{U_2+U_c-c}+\frac{1}{\alpha^2A}\frac{d^2A}{dT^2}\frac{FU_c}{U_2+U_c-c}+J_1F(U_2-c)A_1^{(3)}-F(U_2-c)\frac{1}{\gamma}A_2^{(3)}$$
$$=\gamma F\frac{\Delta}{|\Delta|}-\frac{(a^2+F)}{\alpha^2A}\frac{U_c}{(U_2+U_c-c)^2}\frac{d^2A}{dT^2}+\gamma J_2 \quad \text{(A20)}$$

[substituting in the values of γ from Eq. (5.9)], where

$$J_1=2\int_0^1\sin^2(m\pi y)\left[-\frac{\partial}{\partial y}\left[\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}+F(\Phi_2^{(2)}-\Phi_1^{(2)})\right]-\frac{\partial\Phi_1^{(2)}}{\partial y}\left(\frac{FU_c}{U_2+U_c-c}\right)\right]dy$$
$$=\frac{FU_cm^2\pi^2[|A|^2-|A(0)|^2]}{4(U_2+U_c-c)^2}-\frac{F^2U_c^2m^2\pi^2[|A|^2-|A(0)|^2]}{2(2m^2\pi^2+F)(U_2+U_c-c)^3}$$

$$\times\left[\frac{1}{4}+\frac{\tanh\sqrt{F/2}}{\sqrt{F/2}}\left(\frac{2m^2\pi^2}{2m^2\pi^2+F}\right)\right]$$

$$J_2=2\int_0^1 \sin^2(m\pi y)\left[-\frac{\partial}{\partial y}\left[\frac{\partial^2\Phi_2^{(2)}}{\partial y^2}+F(\phi_1^{(2)}+\Phi_2^{(2)})\right]+\frac{\partial\Phi_2^{(2)}}{\partial y}\left(\frac{FU_c-\eta}{U_2-c}\right)\right]dy$$

$$=-\frac{FU_cm^2\pi^2[|A|^2-|A(0)|^2]}{4(U_2+U_c-c)^2}-\frac{FU_c(FU_c-\eta)m^2\pi^2[|A|^2-|A(0)|^2]}{2(2m^2\pi^2+F)(U_2+U_c-c)^2(U_2-c)}\times\left[\frac{1}{4}+\frac{\tanh\sqrt{F/2}}{\sqrt{F/2}}\left(\frac{2m^2\pi^2}{2m^2\pi^2+F}\right)\right]. \quad \text{(A21)}$$

Note that each of the J has a part with a $(\partial\Phi/\partial y)$ integrand and a part with a $(\partial^3\Phi/\partial y^3)$ integrand. The latter, which we may write as J_k^s, is the manifestation of the changing vorticity gradient:

$$J_1^s=-J_2^s=\frac{m^4\pi^4FU_c[|A|^2-|A(0)|^2]}{(2m^2\pi^2+F)(U_2+U_c-c)^2}\times\left[\frac{1}{2}-\frac{\sqrt{2F}\tanh\sqrt{F/2}}{2m^2\pi^2+F}\right]. \quad \text{(A22)}$$

Combining the two equations, we get

$$C_1\frac{d^2A}{dT^2}+C_2\alpha^2\frac{\Delta}{|\Delta|}A+C_3\alpha^2A[|A|^2-|A(0)|^2]=0 \quad \text{(A23)}$$

where

$$C_1=-\frac{F^2cU_c}{(U_2+U_c-c)^2}-\frac{F\gamma U_c(a^2+F)}{U_2+U_c-c} \quad \text{(A24)}$$

$$C_2=-\frac{F^2c^2}{(U_2+U_c-c)^2}+F^2(U_2+U_c-c)\gamma^2 \quad \text{(A25)}$$

$$C_3[|A|^2-|A(0)|^2]=F(U_2-c)J_1+F(U_2+U_c-c)\gamma^2J_2. \quad \text{(A26)}$$

Using the values of c, U_c and γ, we can simplify this to

$$\frac{d^2A}{dT^2}-c_i^2\alpha^2A+N\alpha^2A[|A|^2-|A(0)|^2]=0 \quad \text{(A27)}$$

where

$$c_i^2=\frac{\eta}{a^2}\cdot\frac{F^{3/2}}{(a^2+2F)^{3/2}} \quad \text{(A28)}$$

$$N=-\frac{m^4\pi^4}{2(2m^2\pi^2+F)}\left(\pm\frac{2\sqrt{x+2}}{x}+\frac{3}{x}\right)+\frac{m^2\pi^2}{4(2m^2\pi^2+F)}(\pm\sqrt{x+2}+2)+\frac{2\sqrt{2F}m^4\pi^4\tanh\sqrt{F/2}}{(2m^2\pi^2+F)^2}\left(2+\frac{3}{x}\pm\frac{(x+2)^{3/2}}{x}\right). \quad \text{(A29)}$$

(Note: $x=a^2/F$).

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