ZENO'S WALK: A RANDOM WALK WITH REFINEMENTS

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ABSTRACT. A self-modifying random walk on $\mathbb{Q}$ is defined from an ordinary random walk on the integers by interpolating a new vertex into each edge as it is crossed. This process converges almost surely to a random variable which is totally singular with respect to Lebesgue measure, and which is supported on a subset of $\mathbb{R}$ having Hausdorff dimension less than 1. By generating function techniques we then calculate the exponential rate of convergence of the process to its limit point, which may be taken as a bound for the convergence of the measure in the Wasserstein metric.

The process may also be described as a random dynamical system, or as a random walk on the space of monotone piecewise linear functions, where moves are taken by successive compositions with a randomly chosen such function.

1. INTRODUCTION

Following a model introduced by Coppersmith and Diaconis [4], a fair amount of work (cf. Davis [5], Pemantle [8]) has been devoted to analyzing reinforced random walks, that is, random walks on graphs where the probability of moving to a given vertex or over a given edge is a function of the number of times that vertex or edge has been selected in the past. Often the reinforcement is supposed to be positive; that is, the more often an edge has been crossed in the past, the more likely it is to be chosen in the future. Among other results, Davis has shown that all positively reinforced random walks on the integers are recurrent to 0, as long as the weights increase moderately

Allowing negative reinforcement complicates the analysis. One approach, discussed at greater length in section 6.4, is to redefine the negative reinforcement as a process of refinement, whereby each step modifies the graph itself, rather than some weights attached to the graph. As a method solving problems about reinforced random walk this has so far proved nearly useless. The natural negatively reinforced random walks are translated into fairly unnatural and intractable random walks with refinement. Zeno's walk, the topic to be examined in this paper, arose from the search for a process that would sit more congenially in this new context. It could be viewed as a particularly radical sort of negative reinforcement process, since an edge once crossed is likely never to be crossed again in its entirety – it is trapped instead on an infinity of intermediate points, crossing and recrossing an ever-shrinking span. It is this tendency that evoked the name "Zeno's Walk".

The process is defined precisely in Section 2. It begins as an ordinary random walk on $\mathbb{Z}$ originating at 0, with independent transition probabilities p (for $+1$) and $q = 1-p$ (for -1). At each step, though, a new vertex is introduced at the midpoint

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of the edge that has just been crossed. A return to the origin thus requires two steps back. An example of four steps in this procedure, going $++-+$:

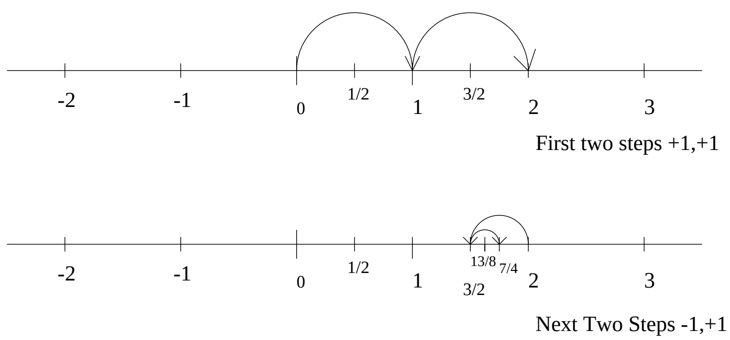

Figure 1

X_n, the location of the process at time n is not at all Markovian: it depends quite strongly upon the order of previous moves. (We note in section 6.3, that the process does have the same marginal distributions as a simple Markov process, but do not make direct use of the fact in this paper.) Suppose, though, that we fix any real number x, and let Y_n be the number of steps to the left from X_n to x. (If $X_n < x$, we let Y_n be negative.) The process Y_n is much simpler than X_n: it is a random walk which steps left or right with probability q or p, making jumps of size 1 when moving toward 0, and of size 2 away from 0. Given any point x, if $\frac{1}{3} < p < \frac{2}{3}$ there is a positive probability that the process will never reach x (Lemma 3.1). It is not surprising, then, that under these conditions the process converges to a real number almost surely (Theorem 1).

What is perhaps unexpected is that the distribution of this limit point may be computed exactly. Here we make use of the essentially stationary nature of the process. While the number values on the points change, the structure of the graph always remains that of $\mathbb{Z}$. If we look at the process at any given time n, and ask what is the probability that it will converge between, say, the next point to the right of X_n and the next one after that, this probability is always the same, no matter what those points are. (The precise statement is given in Lemma 4.2.) This property may be applied recursively to compute the probability of convergence within any desired interval. The result is that the integer part of the limit point is geometrically distributed, while the base-4 digits of the fractional part are i.i.d. variables, dependent only upon the sign of the integer part. (An exact expression for the distribution is given in Corollary 4.6.) This measure is singular with respect to Lebesgue measure, and we can calculate the Hausdorff dimension associated with it by a theorem of Billingsley. The dimension turns out to have a maximum value approximately .9594, when $p = q = \frac{1}{2}$.

In Section 5 we use a generating function to compute the expected distance between the position at time n and the limit point. This expected distance converges geometrically with ratio $(r_p^*)^{-1} = \sqrt[3]{\frac{27pq(p\vee q)}{4}}$. This serves also as an upper bound on the Wasserstein distance between the metrics.

Section 6 suggests variations on this problem which have not been investigated. Most broad of these is 6.3, which sets this problem into the context of compositions of random functions.

2. Definitions

Let $\mathbb{D}$ be the set of dyadic rationals (i.e., $\left\{\frac{n}{2^k} : n, k \in \mathbb{Z}\right\}$).

p and q will always represent nonnegative real numbers whose sum is 1.

Let P be the measure on $\Omega = \{-1, +1\}^{\mathbb{N}}$ by which $\{\omega_n\}_{n=1}^{\infty}$ for $\omega \in \Omega$ are i.i.d. variables with $P\{\omega_n = -1\} = q$, $P\{\omega_n = +1\} = p = 1 - q$. Given $\omega \in \Omega$ we define the $\mathbb{D}$-valued random variables $X_n^{(\omega)}$ and $\mathcal{P}(\mathbb{D})$-valued random variables $S_n^{(\omega)}$ as follows:

i) $X_0^{(\omega)} = 0$.

ii) $X_{n+1}(\omega) = \begin{cases} \min\{x \in S_n^{(\omega)} : x > X_n^{(\omega)}\} & \text{if } \omega_n = +1, \\ \max\{x \in S_n^{(\omega)} : x < X_n^{(\omega)}\} & \text{if } \omega_n = -1. \end{cases}$

iii) $S_{n+1}^{(\omega)} = S_n^{(\omega)} \cup \left\{ \dfrac{X_n^{(\omega)} + X_{n+1}^{(\omega)}}{2} \right\}$.

This defines an injection $\omega \to X^{(\omega)}$ mapping paths on $\mathbb{Z}$ to paths on $\mathbb{Q}$ in which a new vertex is introduced at the midpoint of an edge as it is crossed. $S_n^{(\omega)}$ is the set of vertices at time n.

Given $x_1, x_2 \in \mathbb{R}$ and $n \in \mathbb{N}_0$ define

$$d_n^{(\omega)}(x_1, x_2) = \begin{cases} \#\left\{y \in S_n^{(\omega)} : x_1 \le y < x_2\right\} & \text{if } x_1 \le x_2, \\ -\#\left\{y \in S_n^{(\omega)} : x_2 \le y < x_1\right\} & \text{if } x_2 < x_1. \end{cases}$$

So $d_n^{(\omega)}(x_1, x_2)$ is the (signed) distance between the rational points x_1 and x_2 at time n, measured in the number of vertices.

Given $a \in \mathbb{Z}$ and $m \in \mathbb{N}_0$, define $\sigma_m^{(\omega)}(a)$ to be the unique element of $S_m^{(\omega)}$ at distance a from $X^{(\omega)}$ at time m. Note that $d_m(X_m^{(\omega)}, \sigma_m^{(\omega)}(a)) = a$. Thus,

$$S_m^{(\omega)} = \{\sigma_m^{(\omega)}(a) : a \in \mathbb{Z}\}$$

with

$$\cdots < \sigma_m^{(\omega)}(-2) < \sigma_m^{(\omega)}(-1) < \sigma_m^{(\omega)}(0) = X_m^{(\omega)} < \sigma_m^{(\omega)}(1) < \cdots .$$

We define $\Delta_m^{(\omega)}(a) = \sigma_m^{(\omega)}(a+1) - \sigma_m^{(\omega)}(a)$.

To reduce the clutter of notation, the superscript (ω) will generally be dropped, except where the dependence on ω needs to be emphasized.

3. Convergence

Let $x \in S_n$ be fixed. (Formally, we may let x be a random variable which is measurable with respect to $X_1, \dots, X_n$ and contained in S_n.) Let $Y_m = d_{m+n}(x, X_{m+n})$. Then Y_m is a Markov process on $\mathbb{Z}$.

$$\text{If } Y_m > 0,\ Y_{m+1} = Y_m + \begin{cases} 2 \text{ with prob. } p, \\ -1 \text{ with prob. } q. \end{cases} \tag{1}$$

$$\text{If } Y_m < 0,\ Y_m = Y_m + \begin{cases} 1 \text{ with prob. } p, \\ -2 \text{ with prob. } q. \end{cases} \tag{2}$$

$$\text{If } Y_m = 0,\ Y_{m+1} = \begin{cases} 2 \text{ with prob. } p, \\ -2 \text{ with prob. } q. \end{cases} \tag{3}$$

That is, Y, the distance of the process from a fixed point, is a random walk which takes steps right or left with probability p or q respectively, where steps away from zero have length 2 and steps toward zero have length 1.

Lemma 3.1. *$P\{\exists m \ge 0$ s.t. $Y_m = 0\} =$*

i) *1 if $Y_0 \le 0$ and $p \ge \frac{2}{3}$,*
ii) *1 if $Y_0 \ge 0$ and $p \le \frac{1}{3}$,*
iii) *$\alpha_p^{|Y_0|}$ if $Y_0 \le 0$ and $p < \frac{2}{3}$, where $\alpha_p = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{4p}{q} + 1}$*
iv) *$\alpha_q^{Y_0}$ if $Y_0 \ge 0$ and $p > \frac{1}{3}$ where $\alpha_q = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{4q}{p} + 1}$.*

(Note that $0 < \alpha_p < 1$ for $p \in (0, \frac{2}{3})$, and $\alpha_{\frac{2}{3}} = 1$; similarly for α_q.)

Proof. The arguments are standard.

i) Let Y' be the process Y stopped at the time of first arrival at 0. We have

$$\begin{aligned} E(Y'_{m+1} \big| Y'_1, \ldots, Y'_m) &= \begin{cases} Y'_m + 2q - p & \text{if } Y'_m < 0, \\ 0 & \text{if } Y'_m = 0 \end{cases} \\ &\ge Y'_m, \end{aligned}$$

so Y' is a negative submartingale, hence converges almost surely to 0.

ii) Same as above, but with positive supermartingale in place of negative submartingale.

iii) Note that α_p is a root of the polynomial $qx^3 - x + p = q(x-1)(x^2 + x - \frac{p}{q})$. Define $Y''_m = \alpha_p^{|Y'_m|}$. Then

$$\begin{aligned} E(Y''_{m+1} \big| Y''_1, \ldots, Y''_m) &= \begin{cases} Y''_m\big(p \cdot \alpha_p^{-1} + q \cdot \alpha_p^2\big) & \text{if } Y'_m < 0, \\ 1 & \text{if } Y'_m = 0 \end{cases} \\ &= Y''_m. \end{aligned}$$

since $p \cdot \alpha_p^{-1} + q \cdot \alpha_p^2 = 1$. Y'' is a bounded martingale, hence converges almost surely, and

$$\begin{aligned} \alpha_p^{|Y_m|} &= E(Y''_0) \\ &= E\left(\lim_{m\to\infty} Y''_m\right) \\ &= 1 \cdot P\big\{Y \text{ reaches } 0\big\} + 0 \cdot P\big\{Y \text{ goes to } -\infty\big\} \\ &= P\big\{Y \text{ reaches } 0\big\}. \end{aligned}$$

iv) Same as above, exchanging p and q. □

Corollary 3.2. *Let μ be a stopping time for (X), $\kappa \in \mathbb{Z}$ fixed. Then the probability that the process, beginning from time μ, will ever reach the point κ to the right, $P\{\exists m \ge \mu$ s.t. $X_m = \sigma^\kappa(\mu) | \mu < \infty\} =$*

i) *1 if $\kappa \ge 0$ and $p \ge \frac{2}{3}$,*
ii) *1 if $\kappa \le 0$ and $p \le \frac{1}{3}$,*
iii) *$(\alpha_p)^\kappa$ if $\kappa \ge 0$ and $p < \frac{2}{3}$,*
iv) *$(\alpha_q)^{|\kappa|}$ if $\kappa \le 0$ and $p > \frac{1}{3}$.*

Theorem 1. *i) If* $p \leq \frac{1}{3}$ *then* $\lim_{n\to\infty} X_n = -\infty$ *P-a.s.*

ii) If $p \geq \frac{2}{3}$ *then* $\lim_{n\to\infty} X_n = +\infty$ *P-a.s.*

iii) If $\frac{1}{3} < p < \frac{2}{3}$ *then* $X_\infty = \lim_{n\to\infty} X_n$ *exists and is finite P-a.s.*

Proof. Let $T_x = \{X_n = x \text{ for infinitely many n}\}$ for any $x \in \mathbb{D}$. The probability of T_x is just the probability that the Markov process Y defined in (1) is recurrent to 0. Since the process is irreducible, this probability is independent of the starting point, consequently independent of x, and is either 0 or 1. We have

$$\begin{aligned} P(T_x) &\leq P\{Y_n = 0 \text{ for some } n > 0 \mid Y_0 = 0\} \\ &= p \cdot P\{Y_n = -2 \text{ for some } n > 1\} + q \cdot P\{Y_n = +2 \text{ for some } n > 1\} \\ &= \begin{cases} p \cdot 1 + q \cdot \alpha_p^2 & \text{if } p \leq \frac{1}{3}, \\ p \cdot \alpha_q^2 + q \cdot 1 & \text{if } p \geq \frac{2}{3}, \\ p \cdot \alpha_p^2 + q \cdot \alpha_q^2 & \text{if } \frac{1}{3} < p < \frac{2}{3} \end{cases} \\ &< p + q = 1. \end{aligned}$$

Since $P(T_x) < 1$ it must be 0.

Now choose any dyadic rational x. We have

$$\big\{\liminf_{n\to\infty} X_n < x < \limsup_{n\to\infty} X_n\big\} \subset \big\{X_n = x \text{ infinitely often}\big\},$$

so

$$P\big\{\liminf_{n\to\infty} X_n < x < \limsup_{n\to\infty} X_n\big\} = 0.$$

Taking a union over all x, we get

$$P\big\{\liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n\big\} = 0.$$

Thus X_n converges almost surely to a (possibly infinite) random variable X_∞.

If $\frac{1}{3} < p < \frac{2}{3}$,

$$P\big\{|X_\infty| = \infty\big\} \leq \lim_{N\to\infty} P\big\{\exists m \text{ s.t. } |X_m| = N\big\} \leq \lim_{N\to\infty} \alpha_p^N + \alpha_q^N = 0.$$

which proves (iii).

If $p \leq \frac{1}{3}$, given any positive integer N,

$$P\big\{X_n < -N \text{ infinitely often}\big\} = 1$$

by Corollary 3.2. This means that, the only possible limit for X_n is $-\infty$, proving (i). Similarly, if $p \geq \frac{2}{3}$, $X_\infty = +\infty$, proving (ii). □

4. The limiting measure

We assume from now on that $\frac{1}{3} < p < \frac{2}{3}$, and restrict attention to the subspace of Ω on which $X^{(\omega)}$ converges.

To calculate the distribution of $X_\infty^{(\omega)}$ we use the fact that if the process is stopped at any time, it then runs forward exactly as from the start, only on a new lattice. To make this precise, we define a random piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ by

$$f_n^{(\omega)}(x) := \sigma_n^{(\omega)}(\lfloor x \rfloor) + (x - \lfloor x \rfloor)\Delta_n^{(\omega)}(\lfloor x \rfloor).$$

In particular, $X_n = f_n(0)$.

Also define

$$f^+(x) := \begin{cases} x+1 & \text{if } x \geq 0, \\ \frac{x}{2}+1 & \text{if } 0 \geq x \geq -2, \\ x+2 & \text{if } -2 \geq x; \end{cases}$$

$$f^-(x) := \begin{cases} x-1 & \text{if } x \leq 0, \\ \frac{x}{2}-1 & \text{if } 0 \leq x \leq 2, \\ x-2 & \text{if } 2 \leq x. \end{cases}$$

f^+ and f^- correspond to f_1 in the cases $\omega_1 = +1$ and $\omega_1 = -1$ respectively. The essential fact is

Lemma 4.1.

$$f_n^{(\omega)} = f^{\omega_1} \circ f^{\omega_2} \circ \cdots \circ f^{\omega_n}.$$

Proof. We perform induction on n. For $n = 1$ the statement is trivial. Assume it now true for n. We need to show that for all real numbers x,

$$f_{n+1}^{(\omega)}(x) = f_n^{(\omega)}(f^{\omega_{n+1}}(x)). \tag{4}$$

In fact, we only need to establish this for integers x, since the functions f are linear in between. Consider the case $\omega_{n+1} = +1$. Then for integers x,

$$f_{n+1}(x) = \sigma_{n+1}(x) = \begin{cases} \sigma_n(x+1) & \text{if } x \geq 0, \\ \frac{\sigma_n(0)+\sigma_n(1)}{2} & \text{if } x = -1, \\ \sigma_n(x+2) & \text{if } x \leq -2. \end{cases} \tag{5}$$

Since by linearity $f_n(\frac{1}{2}) = \frac{\sigma_n(0)+\sigma_n(1)}{2}$, (5) is equivalent to (4). We can then do the same for the case $\omega_{n+1} = -1$ to complete the proof. □

Lemma 4.2. *Let τ be a stopping time for X such that $P\{\tau < \infty\} > 0$, S any subset of $\mathbb{R}$. Then*

$$P\{X_\infty \in S\} = P\{X_\infty \in f_\tau(S) \big| \tau < \infty\}. \tag{6}$$

This follows from the finite statement:

Lemma 4.3. *Let τ be a stopping time such that $P\{\tau < \infty\} > 0$, S any subset of $\mathbb{R}$, n a nonnegative integer. Then*

$$P\{X_n \in S\} = P\{X_{n+\tau} \in f_\tau(S) \big| \tau < \infty\}. \tag{7}$$

Proof. On the set $\{\tau < \infty\}$, let ω be shifted τ places to the left to form a new sequence ω^*; that is, $\omega_n^* = \omega_{\tau+n}$. Then by Lemma 4.1 and by the invertibility of all the functions f on the restriction of ω to the set $\{\tau^{(\omega)} < \infty\}$,

$$\begin{aligned} \{\omega :\ X_{n+\tau}^{(\omega)} \in f_\tau^{(\omega)}(S)\} &= \{\omega : f_{n+\tau}^{(\omega)}(0) \in f_\tau^{(\omega)}(S)\} \\ &= \{\omega : f_\tau^{(\omega)}\big(f_n^{(\omega^*)}(0)\big) \in f_\tau^{(\omega)}(S)\}. \\ &= \{\omega : f_n^{(\omega^*)}(0) \in S\} \end{aligned}$$

Since the ω_i are independent,

$$\begin{aligned} P\{X_{n+\tau}^{(\omega)} \in f_\tau^{(\omega)}(S) \text{ and } \tau < \infty\} &= P\{f_n^{(\omega^*)}(0) \in S \text{ and } \tau < \infty\} \\ &= P\{f_n^{(\omega)}(0) \in S\} \cdot P\{\tau < \infty\}. \end{aligned}$$

□

Given $S \subset \mathbb{R}$ we will write $P_\infty(S)$ for $P\{X_\infty \in S\}$.

Corollary 4.4. *Let $N \in \mathbb{N}_0$, $r_1, r_2 \in \mathbb{R}$, $0 \le r_1 \le r_2$. Then*

$$P_\infty([N+r_1, N+r_2]) = P\{X \text{ reaches } N\} \cdot P_\infty([r_1, r_2]) = \alpha_p^N \cdot P\{[r_1, r_2]\}, \tag{8}$$

and

$$P_\infty\{[-N-r_2, -N-r_1]\} = \alpha_q^N \cdot P_\infty([-r_2, -r_1]). \tag{9}$$

Proof. We make the choice $\tau = \inf\{n : X_\omega = N\}$ in Lemma 4.2 □

In calculating the distribution P_∞, it will be convenient to be able to elide the distinction between open and closed intervals; this is justified by the following

Lemma 4.5. *The measure P_∞ is non-atomic; that is, for all $x \in \mathbb{R}$, $P_\infty(\{x\}) = 0$.*

Proof. First, we have

$$P\{X_\infty = 0\} \le P\{Y \text{ recurrent}\}.$$

where Y is the Markov process defined in (1); the latter probability is 0 by Lemma 3.1. By Lemma 4.2, $P\{X_\infty = x \;\&\; \exists\tau \text{ s.t. } X_\tau = x\} = 0$ for all $x \in \mathbb{D}$. It remains then to show that

$$\forall x \in \mathbb{R}, \quad P\{\forall n \in \mathbb{N},\ X_n \ne x \text{ and } \lim_{n\to\infty} X_n = x\} = 0.$$

Let ω be in this set. We must have

$$\lim_{n\to\infty} \operatorname{dist}(x, S_n^{(\omega)}) = 0.$$

Since this distance can decrease only when the process skips over x, this means that if we define stopping times

$$\tau^{(0)} = \rho^{(0)} = 0,$$
$$\tau^{(i+1)} = \inf\{n > \rho^{(i)} : X_n > x\},$$
$$\rho^{(i+1)} = \inf\{n > \tau^{(i+1)} : X_n < x\},$$

we have

$$\{X_\infty = x\} \subset \{\tau^{(1)} < \rho^{(1)} < \tau^{(2)} < \rho^{(2)} < \cdots < \infty\}.$$

Lemma 3.1 implies that

$$P\{\tau^{(i)} < \infty \mid \rho^{(i-1)} < \infty\} \le \alpha_p, \text{ and}$$
$$P\{\rho^{(i)} < \infty \mid \tau^{(i)} < \infty\} \le \alpha_q.$$

Thus the probability that all the stopping times are finite is 0. □

Theorem 2. *Let $r = \sum_{i=1}^k 4^{-i} \cdot a_i$, where $a_i \in \{0,1,2,3\}$. (That is, $r = 0.a_1a_2\ldots a_k$ is the base-4 representation of r.) Let c_i $(i = 0,1,2,3)$ be the number of digits equal to i. Then*

i) $P_\infty([r, r+4^{-k}]) = \alpha_p^{c_0+2c_1+c_2+2c_3} \cdot \alpha_q^{2c_0+2c_1+c_2+c_3} \cdot P_\infty([0,1])$,

ii) $P_\infty([-r-4^{-k}, -r]) = \alpha_q^{c_0+2c_1+c_2+2c_3} \cdot \alpha_p^{2c_0+2c_1+c_2+c_3} \cdot P_\infty([-1,0])$,

iii) $P_\infty([-1,0]) = \frac{(1-\alpha_p)(1-\alpha_q)}{1-\alpha_p\alpha_q^2}$,

iv) $P_\infty([0,1]) = \frac{(1-\alpha_p)(1-\alpha_q)}{1-\alpha_q\alpha_p^2}$.

Proof. The idea here is simply that, in order to wind up in a given interval of length 4^{-k}, first the process must make the necessary 2, 3, or 4 moves that will chop out the desired interval of length $\frac{1}{4}$ (regardless of where else it might wander betweentimes), and then wind up in the right portion of this quarter. Consider, for instance, the probability of ending up in $[\frac{4}{16}, \frac{5}{16}]$. The process must first make its way to 1, which has probability α_p of ever happening. Then it must find its way back to 0, which has probability α_q^2. Then it must reach $\frac{1}{4}$, at which point the interval $[0,1]$ will have the following structure:

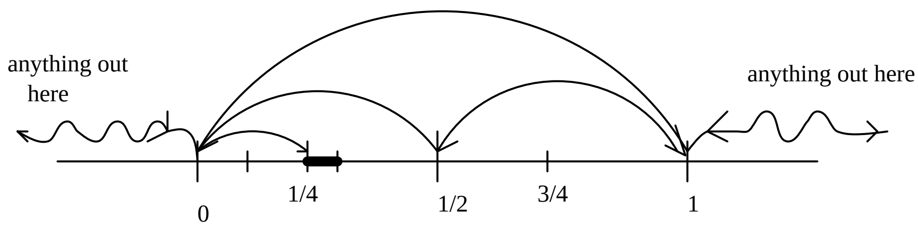

Figure 2

The probability of ever finding the process in such a configuration is $\alpha_p^2 \alpha_q^2$. Once there, the probability of ending in $[\frac{4}{16}, \frac{5}{16}]$ is the same as $P_\infty([0, \frac{1}{4}])$, by Lemma 4.2. Formally, we will perform induction on k. Let

$$A(c_0, c_1, c_2, c_3) = \alpha_p^{c_0+2c_1+c_2+2c_3} \cdot \alpha_q^{2c_0+2c_1+c_2+c_3}.$$

i) For $k = 0$ the statement is trivial. Assume true for $k-1$. Let

$$\begin{aligned}
\tau &= \inf\{n : X_n = 1\}, \\
\tau' &= \inf\{n > \tau : X_n = 0\}, \\
\tau'' &= \inf\{n : X_n = \frac{1}{2}\}, \\
r' &= 0.a_2 a_3 \ldots a_k = 4r - a_1.
\end{aligned}$$

Consider two cases:

Case 1: $a_1 = 0$ or 1. On the set $\{\omega : \tau' < \infty\}$, $f_{\tau'}^{(\omega)} = \frac{x}{4}$ for $0 \le x \le 2$. Further,

$$\{\omega : X_\infty^{(\omega)} \in (0, \frac{1}{2})\} \subset \{\omega : \tau' < \infty\}.$$

We then have by Lemma 4.2,

$$P_\infty\big([r, r+4^{-k}]\big) = P\{\tau' < \infty\} \cdot P_\infty\big([4r, 4r + 4^{-k+1}]\big).$$

By the induction hypothesis and Corollary 4.4,

$$\begin{aligned}
P_\infty\big([4r, 4r+4^{-k+1}]\big) &= P_\infty\big([0,1]\big) \cdot \alpha_p^{a_1} \cdot \begin{cases} A(c_0 - 1, c_1, c_2, c_3) & \text{if } a_1 = 0, \\ A(c_0, c_1 - 1, c_2, c_3) & \text{if } a_1 = 1 \end{cases} \\
&= \alpha_p^{-1} \alpha_q^{-2} \cdot A(c_0, c_1, c_2, c_3) \cdot P_\infty\big([0,1]\big).
\end{aligned}$$

Since

$$\begin{aligned}
P\{\tau' < \infty\} &= P\{X \text{ reaches } \sigma^1\} \cdot P\{X \text{ reaches } \sigma^{-2}\} \\
&= \alpha_p \cdot \alpha_q^2 \qquad \text{by Corollary 3.2,}
\end{aligned}$$

this gives us the result (i).

Case 2: $a_1 = 2$ or 3. Exactly as above, only with τ' replaced by τ''.

$$P_\infty\big([r, r+4^{-k}]\big) = P\{\tau'' < \infty\} \cdot P_\infty\big([4r-2, 4r-2+4^{-k+1}]\big).$$

$$P\big([4r-2, 4r-2+4^{-k+1}]\big) = P_\infty\big([0,1]\big) \cdot \begin{cases} A(c_0, c_1, c_2-1, c_3) & \text{if } a_1 = 2, \\ \alpha_p \cdot A(c_0, c_1, c_2, c_3-1) & \text{if } a_1 = 3, \end{cases}$$

which reduces to

$$\alpha_p^{-1}\alpha_q^{-1} \cdot P_\infty\big([0,1]\big) \cdot A(c_0, c_1, c_2, c_3).$$

Applying now

$$P\{\tau'' < \infty\} = \alpha_p \cdot \alpha_q,$$

we get, again, (i).

ii) Exactly like (i), with p and q interchanged.

iii) We have

$$\begin{aligned} 1 &= \sum_{N=0}^{\infty} \big(P_\infty([N, N+1]) + P_\infty([-N, -N-1])\big) \\ &= \sum_{N=0}^{\infty} \alpha_p^N P_\infty\big([0,1]\big) + \sum_{N=0}^{\infty} \alpha_q^N P_\infty\big([-1,0]\big) \\ &= \frac{P_\infty\big([0,1]\big)}{1-\alpha_p} + \frac{P_\infty\big([-1,0]\big)}{1-\alpha_q}. \end{aligned}$$

Since $X_\infty \in (0,1]$ only if X crosses to 1 at some time,

$$\begin{aligned} P\big([0,1]\big) &= P\{X_\infty \in [0,1] \big| \tau < \infty\} \cdot P\{\tau < \infty\} \\ &= P\{X_\infty \in [\sigma_\tau(-2), \sigma_\tau(0)] \big| \tau < \infty\} \cdot \alpha_p \\ &= P\{X_\infty \in [-2,0]\} \cdot \alpha_p \qquad (10) \\ &= \big(P_\infty([-2,-1]) + P_\infty([-1,0])\big) \cdot \alpha_p \\ &= P_\infty\big([-1,0]\big) \cdot (\alpha_q + 1) \cdot \alpha_p. \end{aligned}$$

This implies that

$$\begin{aligned} 1 &= P_\infty\big([-1,0]\big) \cdot \left[\frac{\alpha_p(\alpha_q+1)}{1-\alpha_p} + \frac{1}{1-\alpha_q}\right] \\ &= P_\infty\big([-1,0]\big) \cdot \frac{1-\alpha_p\alpha_q^2}{(1-\alpha_p)(1-\alpha_q)}. \end{aligned}$$

Thus,

$$P_\infty\big([-1,0]\big) = \frac{(1-\alpha_p)(1-\alpha_q)}{1-\alpha_p\alpha_q^2}.$$

iv) Follows by symmetry from (iii), or from (10) via the identities

$$\alpha_p^2 + \alpha_p = \frac{p}{q}, \qquad \alpha_q^2 + \alpha_q = \frac{q}{p}. \qquad (11)$$

□

If we let

$$X_\infty = sgn \cdot \left(N + \sum_{i=1}^{\infty} 4^{-i} a_i\right)$$

where

$$sgn \in \{+1, -1\},$$
$$N \in \mathbb{N}_0, \text{ and}$$
$$a_i \in \{0, 1, 2, 3\},$$

this result may be more comprehensibly summarized as

Corollary 4.6. *$\{a_i\}$ are i.i.d. variables, independent of N, with*

$$P\{a_i = k \mid sgn = +1\} = \begin{cases} \alpha_q^2 \alpha_p & \text{if } k = 0, \\ \alpha_q^2 \alpha_p^2 & \text{if } k = 1, \\ \alpha_q \alpha_p & \text{if } k = 2, \\ \alpha_q \alpha_p^2 & \text{if } k = 3. \end{cases} \tag{12}$$

$$P\{a_i = k \mid sgn = -1\} = \begin{cases} \alpha_p^2 \alpha_q & \text{if } k = 0, \\ \alpha_p^2 \alpha_q^2 & \text{if } k = 1, \\ \alpha_p \alpha_q & \text{if } k = 2, \\ \alpha_p \alpha_q^2 & \text{if } k = 3. \end{cases} \tag{13}$$

N is geometrically distributed, with

$$P\{N = k \mid sgn = +1\} = (\alpha_p)^k \cdot (1 - \alpha_p) \text{ and} \tag{14}$$

$$P\{N = k \mid sgn = -1\} = (\alpha_q)^k \cdot (1 - \alpha_q). \tag{15}$$

Finally,

$$P\{sgn = +1\} = \frac{1 - \alpha_q}{1 - \alpha_q \alpha_p^2} \text{ and} \tag{16}$$

$$P\{sgn = -1\} = \frac{1 - \alpha_p}{1 - \alpha_p \alpha_q^2}. \tag{17}$$

If we take the symmetric case, in which $p = q = \frac{1}{2}$, the distribution in Corollary 4.6 takes on a particularly simple form:

$$sgn, N, \text{ and } \{a_i\} \text{ are all independent;}$$
$$P\{sgn = +1\} = P\{sgn = -1\} = \frac{1}{2};$$
$$P\{N = k\} = \alpha^{N+2};$$
$$P\{a_i = k\} = \begin{cases} \alpha^3 & \text{if k=0,} \\ \alpha^4 & \text{if k=1,} \\ \alpha^2 & \text{if k=2,} \\ \alpha^3 & \text{if k=3,} \end{cases}$$

where $\alpha = \alpha_{\frac{1}{2}} = \frac{\sqrt{5}-1}{2}$ is the golden mean.

From these it is straightforward to show

Proposition 4.7. *The measure P_∞ is absolutely singular with respect to Lebesgue measure. Further, any two measures P_∞ corresponding to distinct p, q are absolutely singular.*

More generally, we know the following:

Lemma 4.8. *Let (a_i) and (b_i) be two sequences of i.i.d. random variables on $\{0, 1, \ldots, k-1\}$, with*

$$P\{a_i = j\} = p_j \qquad (j = 0, 1, \ldots, k-1) \text{ and}$$
$$P\{b_i = j\} = q_j \qquad (j = 0, 1, \ldots, k-1).$$

Let μ and ν be the distributions of $\sum_{i=1}^{\infty} k^{-i} \cdot a_i$ and $\sum_{i=1}^{\infty} k^{-i} \cdot b_i$ respectively on $[0, 1)$. If for some j, $p_j \neq q_j$, then $\mu \perp \nu$.

Proof. Suppose, without loss of generality, $p_0 \neq q_0$. Given any $x \in [0, 1)$ define $z_n(x) = \#$ of zeroes in the first n digits of the base-k representation of x. Let

$$A = \{x \in [0, 1) : \lim_{n \to \infty} \frac{z(n)}{n} = p_0\}.$$

By the strong law of large numbers $\mu(A) = 1$ while $\nu(A) = 0$. □

Now define the measures $\bar{P}_1$ and $\bar{P}_2$ on $[0, 1)$ by

$$\bar{P}_1(S) = P\{\text{frac. part of } X_\infty \in S \,|X_\infty > 0\} \text{ and}$$
$$\bar{P}_2(S) = P\{\text{frac. part of } |X_\infty| \in S \,|X_\infty < 0\}.$$

Lemma 4.8 tells us that $\bar{P}_1$ and $\bar{P}_2$ are singular with respect to Lebesgue measure. We may ask, then, what is the smallest Hausdorff measure of a set with Lebesgue measure 0 and $\bar{P}_i$-measure 1.

Given $x = \sum_{i=0}^{\infty} a_i 4^{-i} \in [0, 1)$, let $\beta_n(x) \in [0, 1]^4$ be the empirical distribution of $\{a_1, \ldots, a_n\}$ on $\{0, 1, 2, 3\}$, and define

$$M_1 = \{x \in [0, 1) \;:\; \lim_{n \to \infty} \beta_n(x) = (\alpha_q^2 \alpha_p, \alpha_q^2 \alpha_p^2, \alpha_q \alpha_p, \alpha_q \alpha_p^2)\},$$
$$M_2 = \{x \in [0, 1) \;:\; \lim_{n \to \infty} \beta_n(x) = (\alpha_p^2 \alpha_q, \alpha_q^2 \alpha_p^2, \alpha_q \alpha_p, \alpha_p \alpha_q^2)\}.$$

By the strong law of large numbers and Corollary 4.6, M_i has $\bar{P}_i$-measure 1 and Lebesgue measure 0.

Billingsley's article [3] includes the following theorem:

Theorem (Billingsley) . *Given $x = \sum_{i=0}^{\infty} a_i b^{-i}$ $(a_i \in \{0, \ldots, b-1\}$, $b \geq 2\}$, define*

$$u_n(x) = \{x' = \sum_{i=0}^{\infty} a_i' b^{-i} \in [0, 1) \;:\; a_i = a_i' \,\forall i \leq n\}.$$

Let μ and ν be measures on $[0, 1)$. Let $\delta \in [0, 1)$ and $M \subset [0, 1)$ be such that

$$M \subset \left\{x \;:\; \lim_{n \to \infty} \frac{\log \nu(u_n(x))}{\log \mu(u_n(x))} = \delta\right\}.$$

Then

$$\dim_\mu M = \delta \dim_\nu M,$$

where $\dim_\mu M$ is the Hausdorff dimension of M with respect to μ.

(N.B.: Billingsley's theorem refers not to subsets of $\mathbb{R}$ with coverings by open intervals, but to sequences of outcomes of a stochastic process on a finite state-space with coverings by cylinder sets. Section 3 of [2] shows that these notions of dimension are equivalent.)

In our case, letting $M = M_i$, $\mu =$ Lebesgue measure, $\nu = \bar{P}_i$, we can apply the theorem with δ the relative entropy of ν and μ:

$$\begin{aligned}\delta &= H(\nu : \mu)\\ &= \frac{\alpha_q^2\alpha_p \ln \alpha_q^2\alpha_p + \alpha_q^2\alpha_p^2 \ln \alpha_q^2\alpha_p^2 + \alpha_q\alpha_p \ln \alpha_q\alpha_p + \alpha_q\alpha_p^2 \ln \alpha_q\alpha_p^2}{\ln \frac{1}{4}}\\ &= \left(1 + \frac{p}{q}\alpha_q^2\right) |\log_4 \alpha_q| - \left(1 + \frac{q}{p}\alpha_p^2\right) |\log_4 \alpha_p|. \qquad (18)\end{aligned}$$

Thus $\dim M_1 = \dim M_2 = \kappa$, and if M' is any other set with $\nu(M') = 1$, we have

$$\begin{aligned}\dim_\mu(M') &\geq \dim_\mu(M' \cap M_i)\\ &= \delta \dim_\nu(M' \cap M_i)\\ &= \delta.\end{aligned}$$

since any set with positive ν-measure must have ν-dimension 1. Thus we have

Proposition 4.9. *P_∞ is supported on a set of Hausdorff dimension given by* (18), *and on no set of smaller dimension. The maximum dimension on $p \in (\frac{1}{3}, \frac{2}{3})$ is $\approx .9594$, which is attained at $p = q = \frac{1}{2}$.*

Note: This makes precise the intuition that the limit points spread out gradually as the process becomes more asymmetric.

Proof. The only part that needs to be proved is that $p = q = \frac{1}{2}$ yields the maximum dimension. But this follows by setting $q = 1 - p$ into the formula (18) and differentiating with respect to p. □

5. Rate of Convergence of the Measures

For ρ a metric, μ and ν measures on $\mathbb{R}$ the Wasserstein metric with respect to ρ may be defined as

$$d_\rho(\mu, \nu) = \sup_{f \in Lip_1(\rho)} |\mu(f) - \nu(f)|,$$

where

$$Lip_1(\rho) = \{f \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \forall x, y \in \mathbb{R},\ |f(x) - f(y)| < \rho(x, y)\}.$$

The Wasserstein distance is also [9]

$$d_\rho(\mu, \nu) = \inf_{\substack{X \sim \mu\\ Y \sim \nu}} \{E(\rho(X, Y))\}.$$

where the infimum is taken over all random variables X and Y, defined jointly on any probability space, such that X has law μ and Y has law ν. If we let P and P_n represent the distributions of X_∞ and X_n respectively, then we have the upper bound

$$d(P_n, P) \leq E(|X_n - X_\infty|).$$

This expectation could itself be considered as an interesting measure of the convergence of our process. From here on it is only this quantity, and not the Wasserstein distance *per se*, that we will study.

Since $|X_n - X_\infty| \to 0$ almost surely, and for any $\epsilon < 1$, $E\big(|X_n - X_\infty| \bigm| |X_n - X_\infty| > \epsilon\big)$ is bounded above by $E\big(|X_\infty|\big)+2$, which is finite,

$$\lim_{n\to\infty} E\big(|X_n - X_\infty|\big) = 0 \qquad (19)$$

From Lemma 4.2 we have

$$
\begin{aligned}
&E(|X_n - X_\infty| \mid X_1,\dots,X_n)\\
&\quad = E(|\sigma_n(0) - X_\infty| \mid X_1,\dots,X_n)\\
&\quad = \sum_{i=0}^{\infty} P\big\{X_\infty \in [\sigma_n(i),\sigma_n(i+1)]\big\}\cdot\\
&\qquad\qquad \Big(\sigma_n(i) - \sigma_n(0) + E\big[X_\infty - \sigma_n(i) \mid X_\infty \in [\sigma_n(i),\sigma_n(i+1)]\big]\Big)\\
&\qquad + \sum_{i=0}^{\infty} P\big\{X_\infty \in [\sigma_n(-i-1),\sigma_n(-i)]\big\}\cdot\\
&\qquad\qquad \Big((\sigma_n(0) - \sigma_n(-i)) + E\big[\sigma_n(-i) - X_\infty \mid X_\infty \in [\sigma_n(-i-1),\sigma_n(-i)]\big]\Big)\\
&\quad = k'_p \sum_{i=0}^{\infty} \alpha_p^i \Big[\big(\sigma_n(i) - \sigma_n(0)\big) + \epsilon_p \Delta_n(i)\Big] \;+\; k'_q \sum_{i=0}^{\infty} \alpha_q^i \Big[\big(\sigma_n(0) - \sigma_n(-i)\big) + \epsilon_q \Delta_n(-i-1)\Big]\\
&\quad = k'_p \sum_{j=0}^{\infty} \Delta_n(j)\Big(\epsilon_p \alpha_p^j \;+\; \sum_{i=j+1}^{\infty} \alpha_p^i\Big) \;+\; k'_q \sum_{j=0}^{\infty} \Delta_n(-j-1)\Big(\epsilon_q \alpha_q^j \;+\; \sum_{i=j+1}^{\infty} \alpha_q^i\Big)\\
&\quad = k_p \sum_{i=0}^{\infty} \Delta_n(i)\alpha_p^i \;+\; k_q \sum_{i=0}^{\infty} \Delta_n(-i-1)\alpha_q^i,
\end{aligned}
$$

where k'_p and $\epsilon_p = E_p\big(X_\infty - \lfloor X_\infty \rfloor \big| X_\infty > 0\big)$ are positive constants depending only on p, which may be calculated from the formulae in Corollary 4.6, and

$$k_p = k'_p\Big(\epsilon_p + \frac{\alpha_p}{1-\alpha_p}\Big).$$

(The subscript in E_p always represents the probability of a rightward step.) Then

$$E(|X_n - X_\infty|) = k_p \sum_{i=0}^{\infty} E(\Delta_n(i))\alpha_p^i \;+\; k_q \sum_{i=0}^{\infty} E(\Delta_n(-i-1))\alpha_q^i.$$

Note that $E_p(\Delta(i)) = E_{1-p}(\Delta(-i-1))$, so this becomes

$$E_p(|X_n - X_\infty|) = k_p \sum_{i=0}^{\infty} E_p(\Delta_n(i))\alpha_p^i \;+\; k_q \sum_{i=0}^{\infty} E_q(\Delta_n(i))\alpha_q^i. \tag{20}$$

Define the generating functions

$$f_n^{(p)}(x) = \sum_{i=0}^{\infty} E_p(\Delta_n(i))x^i.$$

Since for any n all but a finite number of coefficients are equal to 1, the series $f_n^{(p)}(x)$ converges precisely for those x with $|x| < 1$. Our upper bound may be restated as

$$d(P_n, P) \le E(|X_n - X_\infty|) = k_p f_n^{(p)}(\alpha_p) + k_q f_n^{(q)}(\alpha_q). \tag{21}$$

Since all intervals have length 1 at time $n = 0$,

$$f_0^{(p)}(x) = \frac{1}{1-x}\,.$$

From the definition of the process we have that

$$\Delta_{n+1}(i) = \begin{cases} \Delta_n(i+1) & \text{if } \omega_n = +1, \\ \Delta_n(i-2) & \text{if } \omega_n = -1 \text{ and } i \geq 2, \\ \frac{1}{2}\Delta_n(-1) & \text{if } \omega_n = -1 \text{ and } i = 0, 1, \end{cases}$$

which yields the recursion

$$\left(f_{n+1}^{(p)}\right)_i = \begin{cases} p\left(f_n^{(p)}\right)_{i+1} + q\left(f_n^{(p)}\right)_{i-2} & \text{if } i \geq 2, \\ p\left(f_n^{(p)}\right)_{i+1} + \frac{q}{2}\left(f_n^{(q)}\right)_0 & \text{if } i = 0, 1. \end{cases} \tag{22}$$

(The subscripts indicate terms in the power series.) Then

$$xf_{n+1}^{(p)}(x) \;=\; (qx^3 + p)f_n^{(p)}(x) + \frac{q}{2}(x^2 + x)f_n^{(q)}(0) - pf_n^{(p)}(0).$$

If we define $g_n^{(p)}(x) = (1-x)f_n^{(p)}(x)$, then $g_0(x) = 1$, and $g_n^{(p)}(x)$ is a polynomial with coefficients in $[0,1]$, satisfying the recursion

$$g_{n+1}^{(p)}(x) = (qx^2 + \frac{p}{x})g_n^{(p)}(x) + \frac{q}{2}(1-x^2)g_n^{(q)}(0) - p\left(\frac{1}{x} - 1\right)g_n^{(p)}(0). \tag{23}$$

In particular,

$$\begin{aligned} g_{n+1}^{(p)}(\alpha_p) &= g_n^{(p)}(\alpha_p) + \frac{q}{2}(1-\alpha_p^2)g_n^{(q)}(0) - p(\frac{1}{\alpha_p} - 1)g_n^{(p)}(0) \\ &= 1 + \sum_{i=0}^{n}\left[\frac{q}{2}(1-\alpha_p^2)g_i^{(q)}(0) - p(\frac{1}{\alpha_p} - 1)g_i^{(p)}(0)\right]. \end{aligned} \tag{24}$$

By (19) and (21) we know that $\lim_{n\to\infty} g_n(\alpha_p) = 0$, so (24) may be restated as

$$g_n^{(p)}(\alpha_p) = \sum_{i=n}^{\infty}\left[p(\frac{1}{\alpha_p} - 1)g_i^{(p)}(0) - \frac{q}{2}(1-\alpha_p^2)g_i^{(q)}(0)\right]. \tag{25}$$

Now define

$$h_p(x,y) = \sum_{n=0}^{\infty} g_n^{(p)}(x)y^n.$$

Since $|g_n^{(p)}(x)| \leq (2n+1)$ for $|x| \leq 1$, this sequence is absolutely convergent for all x, y with $|x| < 1$, $|y| < 1$. Substituting the relations (23), we get

$$\begin{aligned} xh_p(x,y) &= x + y \cdot \sum_{n=0}^{\infty} g_{n+1}^{(p)}(x)y^n \\ &= x + y \cdot \sum_{n=0}^{\infty}\left[(qx^3 + p)g_n^{(p)}(x) + \frac{q}{2}(x - x^3)g_n^{(q)}(0) + p(x-1)g_n^{(p)}(0)\right] \\ &= x + (qx^3 + p)yh_p(x,y) + \frac{q}{2}(x - x^3)yh_q(0,y) + p(x-1)yh_p(0,y), \end{aligned}$$

so

$$(x - qx^3y - py)h_p(x,y) = \frac{q}{2}y(-x^3 + x)h_q(0,y) + py(x-1)h_p(0,y) + x. \tag{26}$$

Proposition 5.1. *Let $F(z)$ represent the hypergeometric function with coefficients $(\frac{1}{2},\frac{1}{2};\frac{3}{2})$, and define*

$$\xi_p(y) = \frac{2}{\sqrt{3q}}\frac{1}{\sqrt{y}}\sin\left(\frac{\sqrt{3}}{2}p\sqrt{q}y^{\frac{3}{2}}F\left(\frac{27}{4}qp^2y^3\right)\right).$$

Then

i) *The radius of convergence of ξ around the point $y=0$ is $r_p = \frac{1}{3}\sqrt[3]{\frac{4}{qp^2}}$.*

ii) *For $p \in \left(\frac{1}{3},\frac{2}{3}\right)$ and $|y| \le \min\{r_p, r_{\frac{1}{2}}\}$ we have $|\xi_p(y)| < 1$.*

iii)

$$h_p(0,y) = \frac{2\xi_p\left(2-\xi_q(\xi_p^2+1)\right)}{py(1-\xi_p)(1-\xi_q)\left(4-(\xi_q^2+\xi_q)(\xi_p^2+\xi_p)\right)}. \tag{27}$$

iv) *$h_p(0,y)$ is analytic at $y=0$ with radius of convergence*

$$r_p^* = r_{\max\{p,q\}} = \min\{r_p, r_q\}.$$

Thus,

$$g_n^{(p)}(0) \le \frac{h_p(0,r_p^*)}{\left(r_p^*\right)^n},$$

and

$$\lim_{n\to\infty}\left(g_n^{(p)}(0)\right)^{-1/n} = r_p^*.$$

Idea - The cubic (30), which is the coefficient of h_p in (26), may be solved for ξ as a function of y whenever the discriminant is nonzero. The discriminant vanishes precisely at the points $y = r_p e^{2\pi ik/r}$. ξ_p is defined to be the unique branch which is analytic at 0. (The explicit form of ξ_p is obtained from Cardano's cubic formula; the hypergeometric function is used in place of the more standard arcsine to make the analyticity apparent.) ξ_p is analytic precisely on the disc $\mathcal{D}_p = \{y : |y| < r_p\}$ and, in fact, is singular at all three possible branch points on the perimeter. We show then that the functional equation for h_p implies (27). Given that the denominator doesn't vanish, we know then that the radius of convergence is at least the minimum of r_p and r_q, since h_p is defined in terms of ξ_p and ξ_q. One would expect, too, that h_p must be singular where they are. This is true, the only catch being that we need to show that h_p does not somehow "undo" the algebraic singularity in ξ_p and ξ_q. (A trivial example would be the function $h(y) = \xi(y)^3$, and $\xi(y) = (1-y)^{1/3}$. ξ is singular at $y=1$, but h is not.)

Proof.

i) We have

$$\xi_p(y) = pyG\left(\frac{27}{4}qp^2y^3\right), \tag{28}$$

where

$$\begin{aligned} G(z) &= \frac{3}{\sqrt{z}}\sin\left(\frac{1}{3}\sqrt{z}F(z)\right) \\ &= \frac{1}{\frac{1}{3}\sqrt{zF^2(z)}}\sin\left(\frac{1}{3}\sqrt{zF^2(z)}\right)F(z). \end{aligned}$$

Since products and compositions of analytic functions are analytic, the result follows from the fact that $\frac{\sin\sqrt{z}}{\sqrt{z}}$ is entire, while $F(z)$ is analytic on the disc $|z| \le 1$.

ii) Taking the modulus of both sides in (28) we get

$$\begin{aligned}|\xi_p(y)| &= p|y| \cdot \left| G\left(\left(\frac{y}{r_p}\right)^3\right)\right| \\ &= \frac{1}{2}|y| \left|\frac{\sin\zeta}{\zeta}\right| |F(z)|,\end{aligned}$$

where $|z| \le 1$ and $\zeta = \frac{1}{3}\sqrt{z}F(z)$. Since the series defining $F(z)$ has only positive coefficients, its maximum value on any disc is attained on the positive real axis, which means that $|F(z)| \le F(1) = \frac{\pi}{2}$. Thus $|\zeta| \le \frac{\pi}{6}$. Since $\left|\frac{\sin\zeta}{\zeta}\right|$ achieves its maximum on the imaginary axis, we have

$$\left|\frac{\sin\zeta}{\zeta}\right| \le \frac{\sinh\frac{\pi}{6}}{\frac{\pi}{6}}.$$

Putting these bounds together we get

$$|\xi(y)| \le \frac{1}{2} r_p \frac{\pi}{2} \frac{\sinh\frac{\pi}{6}}{\frac{\pi}{6}} \le \frac{1}{2}\sqrt[3]{\frac{32}{27}} \frac{\pi}{2} \frac{\sinh\frac{\pi}{6}}{\frac{\pi}{6}} < 1.$$

iii) We note first the identity (cf. [1])

$$\sqrt{y}F(y) = \arcsin(\sqrt{y}) \text{ when } 0 < y < 1 \tag{29}$$

which means that on the positive real axis, using basic trigonometric relations,

$$qy\xi_p(y)^3 - \xi_p(y) + py = 0. \tag{30}$$

For $|y| < 1$ let $x = \xi_p(y)$ in (26). The left side is then 0 by (30), yielding

$$0 = \frac{q}{2}(\xi_p - \xi_p^3) y h_q(0, y) + p(\xi_p - 1) y h_p(0, y) + \xi_p.$$

Substituting q for p we get a parallel equation

$$0 = \frac{p}{2}(\xi_q - \xi_q^3) y h_p(0, y) + q(\xi_q - 1) y h_q(0, y) + \xi_q.$$

Solving simultaneously we see that (27) holds on the punctured disc $\{|y| < 1\} \setminus \{0\}$, and hence on all of $\{|y| < 1\}$. Since the numerator and denominator on the right are analytic, and the denominator is nonzero on $\mathcal{D}_p$, this is analytic and equal to $h_p(0, y)$ on all of $\mathcal{D}_p$.

iv) The functions ξ_p and ξ_q are analytic on $\mathcal{D}_p \cap \mathcal{D}_q$. Since (ii) guarantees that the denominator in (27) is nonzero, h_p too is analytic on this disc. We know then that the radius of convergence is at least r_p^*, and apply the Cauchy estimate (cf. [6]); since $h_p(0, y)$ is represented by a series with only positive real coefficients, it attains its maximum on the positive real axis.

In fact, the radius of convergence is exactly r_p^*, but this is irrelevant to the rate of convergence of the Wasserstein distance. It provides a lower bound only on the convergence of the upper estimate $E\big(|X_n - X_\infty|\big)$. To prove it, we need to show that $h_p(0,y)$ cannot be continued analytically in a neighborhood of $y = r_p^*$. We may lift the map $h_p(0,y)$ to the Riemann surface associated to the cubic (30), and then, using the Inverse Function Theorem, show that analyticity of $h_p(0,y)$ at r_p^* would allow us to extend ξ_p and ξ_{1-p} at that point as well.

□

By (25),

$$\begin{aligned} g_n(\alpha_p) &\le p\big(\alpha_p^{-1}-1\big)\sum_{i=n}^{\infty} g_i^{(p)}(0) \\ &\le \frac{p h_p(0,r_p^*)\big(\alpha_p^{-1}-1\big)}{1-\frac{1}{r_p^*}}\big(r_p^*\big)^{-n}. \end{aligned}$$

We have then immediately by (21)

Theorem 3.

$$d(P_n,P) = O\left(\big(r_p^*\big)^{-n}\right).$$

Thus the convergence is exponential, with its fastest rate corresponding to $r_{\frac12}^{-1} = \sqrt[3]{\frac{27}{32}} \approx .945$ at $p=\frac12$, and slowing down to $r_p^* = 1$ at $p = \frac13$ or $\frac23$, when the process does not converge at all.

6. Open questions

6.1. **Other refinement rules.** Let λ be a probability measure on the set of finite subsets of $(0,1)$. Instead of introducing one new point at the midpoint of each segment that is crossed, we could define the process by introducing a suitably scaled copy of the set Λ chosen from λ. As long as p is chosen so that

$$\max\left\{\frac{p}{q},\frac{q}{p}\right\} < 1+E.$$

where E is the expected size of Λ, the proof of Lemma 3.1 and hence of Theorem 1 go through unchanged, to prove that $X(n)$ converges almost surely. The limiting measure will not, however, take on such an easily calculable form. What can we say about it?

6.2. **Alternate topologies.**

6.2.1. *Trees.* It is hard to imagine a reasonable generalization to general graphs other than $\mathbb{Z}$. However, Agoston Pisztora has pointed out that there are natural analogues to Zeno's walk on trees. The idea is to begin with an initial sapling, which may be a finite or infinite tree, and we are given some probability measure P on the set of pairs (T,v), where T is a tree and v is a vertex of T, its "root". We perform an ordinary random walk on the tree, taking each step to one of the nearest neighbors with equal probability, but now at every step we not only interpolate a vertex, call it x, into the edge that is crossed, we also choose a pair (T,v) according to the distribution P, and afix this new tree to the old one by identifying x with v. Zeno's walk is the case in which P is simply the measure which assigns probability 1 to the trivial (one-point) tree.

There are many questions that may be posed about this model. We could define a metric on the graph as it grows, for instance by declaring every edge of an added tree to have length one-half that of the edge to which it is appended, and ask whether the distance from the starting point converges, and, if so, what is the distribution of the limit distance. We could also ask how the tree structure grows in time, how many branches it has, how many terminal nodes, etc.

6.2.2. $\mathbb{Z}$ *with modifications.* We could have an absorbing point, or two, or one or two reflecting points. These problems are related to each other, and to the problem of the Zeno's walk on the circle $\mathbb{R}/\mathbb{Z}$. In all of these cases, the recursive method which was applied successfully on $\mathbb{Z}$ breaks down, because the structure of the graph changes at every step.

One useful characteristic of all these examples is that the graph resembles $\mathbb{Z}$ ever more as vertices are added. In this way we can define a backwards recursion from the limiting distributions that we know on $\mathbb{Z}$ that allows us at least to approximate various limiting quantities. For instance, for the symmetric Zeno's walk on the circle,

$$P_\infty\big([0,\frac{1}{4}]\big) = P_\infty\big([\frac{3}{4},0]\big) \approx 0.288, \text{ and}$$
$$P_\infty\big([\frac{1}{4},\frac{1}{2}]\big) = P_\infty\big([\frac{1}{2},\frac{3}{4}]\big) \approx 0.212.$$

But this method seems unsuited to deriving any general results.

6.3. **Compositions of random functions.** This may be the most fruitful line for further investigation. Lemma 4.1 allows us to dispose of the whole model of a self-modifying random walk, and think of Zeno's walk simply as a sequence of compositions of i.i.d. random functions, in this case with the successive choices being from a distribution on the two points $\{f^+, f^-\}$. Theorem 1 may be reinterpreted then as the statement that $\lim_{n\to\infty} f_n(0)$ exists, and has the distribution of X_∞ given in Theorem 2. In fact, the almost-sure convergence implies that $f_n(x)$ converges uniformly on compact sets to a constant function, the constant having the distribution of X_∞. We may ask then, for what other distributions on the space of continuous bijections from $\mathbb{R} \to \mathbb{R}$ will the compositions of successive i.i.d. choices converge? When will the limit be almost surely constant?

Nor must we confine our investigations to $\mathbb{R}$. There is, for instance, in this context a fairly straightforward generalization of Zeno's walk to higher dimensions, which goes as follows: Let Q be some measure on S^{n-1}, the unit $(n-1)$-sphere in $\mathbb{R}^n$. Given $x \in S^{n-1}$, define $f^{(x)} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f^{(y)}(x) = \frac{1}{2}y + (x+y)\phi\big(\|x+y\|\big),$$

where

$$\phi(r) = \begin{cases} \frac{1}{2} & \text{if } r \le 1, \\ 1 - \frac{1}{2r} & \text{if } r > 1. \end{cases}$$

When $n = 1$, this reduces to Zeno's walk. Whereas Zeno's walk may be thought of as choosing a random point on S^0, shifting the center to that point, and blowing up the intervening space by a factor of 2, this generalized process consists of choosing a random point on S^{n-1}, shifting the center to that point, and then blowing up the disk centered on the midpoint of the line-segment traversed. It is not clear which, if any, measures Q will cause the process to converge, much less what the limiting measures might be.

One further remark: If we reverse the order of the compositions, so take

$$X_n^* = f^{\omega_n} \circ f^{\omega_{n-1}} \circ \cdots \circ f^{\omega_1}(0),$$

we get a Markov process, of the sort discussed by Kifer in [7]. Since the f^{ω_i} are i.i.d., $X_n^* =_d X_n$. Thus Zeno's walk, though not a Markov process, has the same marginal distributions as one.

6.4. **Random walks with negative reinforcement.** Consider a process defined as follows: We begin with a sequence of positive real numbers $a_1, a_2, \ldots$, and with each edge $(i, i+1)$ of the graph $\mathbb{Z}$ assigned the weight $w_i(0) = 1$. The process begins at $X(0) = 0$. At time n the process is at some point $X(n) = i$; on the right is the edge $(i, i+1)$ with weight $w_i(n)$, and on the left is the edge $(i-1, i)$ with weight $w_{i-1}(n)$. The process then goes left with probability $w_{i-1}(n)/\big(w_{i-1}(n) + w_i(n)\big)$ and right with probability $w_i(n)/\big(w_{i-1}(n) + w_i(n)\big)$, and the weight of the edge that is crossed goes from a_{j-1} to a_j, where j is the number of times the edge has already been crossed. This is exactly the sort of process considered by Davis [5], except that he required that the weights a_i be increasing.

Now consider this refinement process: We perform a symmetric random walk on the integers, but with the added rule that each time the process passes to a point of the original lattice $\mathbb{Z}$, we interpolate a new vertex into the edge that has been crossed. Only arrival in the initial set of vertices triggers this refinement, although the steps of the walk itself treat all vertices, old and new, identically. For example, the sequence $++--$ would bring the process to 1, with one extra vertex between 0 and 1 and two vertices between 1 and 2. If the next two steps are $-+$ we will be back at 1, without adding any further vertices. If the succeeding steps are $+++$, the walk will arrive at 2, and would add a third vertex between 1 and 2.

Let w_i be the reciprocal of the number of vertices on the edge $[i, i+1)$. Then if $X(n) = i$, by the standard theory of random walks

$$P\big\{X(n+1) = i+1\big\} = 1 - P\big\{X(n+1) = i-1\big\} = \frac{w_i}{w_{i-1} + w_i}.$$

Thus, this process is equivalent to the random walk with negative reinforcement, taking weights $a_j = 1/j$. We might reasonably define the distance $d_n(i,j) = \sum_{k=i}^{j-1} 1/w_k(n)$ (when $i < j$) even when these numbers $1/w_k(n)$ are not integers. This is precisely what Davis does in the case of a_j increasing, whereby he proves that $d_n(0, X(n))$ is a supermartingale, so returns to 0 almost surely.

In the case of decreasing a_j, that is, of negative reinforcement, this is not a supermartingale, so there is no such obvious conclusion to be drawn. It may be possible to use the analogy between negative reinforcement and refinement to study these processes further. There is a considerable problem, though, caused by the nonuniformity of the refinement rule: refinement occurs only when reaching vertices of the original lattice, not when reaching the newly interpolated vertices. Zeno's walk is simply the variant in which all vertices, old and new, are created equal.

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References

[1] Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions, chap. 18. New York : Dover 1972.

[2] Billingsley, P.: Hausdorff dimension in probability theory. Ill. J. Math. **4**, 187–209 (1960).

[3] Billingsley, P.: Hausdorff dimension in probability theory II. Ill. J. Math. **5**, 291–298 (1961).

[4] Coppersmith, D., Diaconis, P. Random walk with reinforcement. Unpublished.

[5] Davis, Burgess: Reinforced random walks. Probability Theory and Related Fields. **84**, 203–229 (1990).

[6] Henrici, Peter. Applied and Computational Complex Analysis, v.1. New York: Wiley 1974.

[7] Kifer, Yuri. Ergodic Theory of Random Transformations. Boston - Basel - Stuttgart: Birkhäuser 1986.

[8] Pemantle, R.: Phase transition in reinforced random walk and RWRE on trees, Annals of Probability, **16**, 1229–1241 (1988).

[9] Shorack, G., Wellner, J. *Empirical Processes with Applications to Statistics.* New York: Wiley 1986.

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